

## **Electronics Primer**



# **Electronics Primer**

Rico A. R. Picone

© 2024 Rico A. R. Picone

All rights reserved. No part of this book may be reproduced in any form by any electronic or mechanical means (including photocopying, recording, or information storage and retrieval) without permission in writing from the author.

## Contents

<b>1</b>	<b>Fundamentals</b>	<b>1</b>
1.1	Voltage, Current, Resistance, and All That	1
1.2	Voltage Dividers	7
1.3	Sources	9
1.4	Thévenin's and Norton's Theorems	12
1.5	Output and Input Resistance and Circuit Loading	14
1.6	Capacitors	16
1.7	Inductors	17
1.8	Problems	18
<b>2</b>	<b>Circuit Analysis</b>	<b>21</b>
2.1	Sign Convention	21
2.2	Methodology for Analyzing Circuits	23
2.3	A Sinusoidal Input Example	25
2.4	Transient and Steady-State Response	29
2.5	Problems	32
<b>3</b>	<b>Steady-State Circuit Analysis for Sinusoidal Inputs</b>	<b>37</b>
3.1	Complex or Phasor Representations of Voltage and Current	37
3.2	Impedance	40
3.3	Methodology for Impedance-Based Circuit Analysis	43
3.4	Voltage and Current Dividers	46
3.5	Problems	49
<b>4</b>	<b>Nonlinear and Multiport Circuit Elements</b>	<b>53</b>
4.1	Transformers	54
4.2	Diodes	56
4.3	MOSFETs	63

4.4	Operational Amplifiers	67
4.5	Problems	71
Bibliography		81

# 1 Fundamentals



In this chapter, we consider fundamental circuit quantities (e.g., voltage and current), topologies (e.g., series and parallel), theorems (e.g., Kirchhoff's laws), and elements (e.g., resistors, capacitors, and inductors).

## 1.1 Voltage, Current, Resistance, and All That

Two quantities will be of special importance in analyzing and designing electronic systems: **voltage** and **current**. The relationship between them defines a third important quantity: **resistance** (more generally, *impedance*).

Momentarily, we will define each of these, but we start with the fundamental quantity in electronics.

### Definition 1.1

Electric charge (or simply charge) is a property of matter that describes the attractive or repulsive force acting on the matter in an electric field. At the microscopic level, charge is quantized into charges of subatomic particles such as protons and electrons, which have opposite charges  $e$  and  $-e$ , where  $e$  is the elementary charge.

Charge has derived SI unit **coulomb** with symbol C. It is considered to be a **conserved quantity**.

### 1.1.1 Voltage

#### Definition 1.2

Voltage is the difference in electrical potential energy of a unit of charge moved between two locations in an electrical field.

Voltage is typically given the variable  $v$  and has derived SI unit **volt** with symbol V.

Voltage is always defined by referring to *two* locations. Sometimes one of these locations is implicitly **ground**—an arbitrarily-defined reference (datum) voltage considered to have zero electrical potential energy—such that we can talk about the voltage “at this” or “at that” location by implicit reference to ground. It is good form to describe the voltage as being “between” two locations or “across” an element.

### 1.1.2 Current

#### Definition 1.3

Current is a flow of charge.

Current is typically denoted  $i$  and has derived SI unit **ampere** with symbol A.

We typically *generate* voltage by doing work on charges. Conversely, we *get* currents by placing voltage across matter through which current can flow. This implies that voltage causes current. Causality here is quite complex, but I will posit the following proposition. We typically observe current when applying voltage, so from a phenomenological point-of-view, it is natural to consider voltage causal of current.<sup>1</sup>

### 1.1.3 Circuits

Electric **circuits** are dynamic electrical systems in which charge accumulates in and flows through elements. Circuit elements are connected via metallic conductors called **wires**, which ideally have the same voltage (relative to, say, ground) everywhere.

### 1.1.4 Circuit Topology

A circuit has a few basic topological features.

A circuit **node** is a continuous region of a circuit that has the same voltage everywhere. A node is an idealized concept that is approximate in most instantiations.

A circuit **element** is a region of a circuit considered to have properties distinct from the surrounding circuit. Examples of elements are resistors, capacitors, inductors, and sources.

A circuit element has **terminals** through which it connects to a circuit.

Circuit elements in **parallel** are those that have two terminals, each of which is shared by another element’s two terminals.

Circuit elements in **series** are those that have two terminals, only one of which is shared between them and this one cannot be shared with any other element.

1. Note that subtlety emerges not only when considering fields, small distances, and short durations—it also arises when we consider certain circuit elements that exhibit behavior related to the time rate of change of voltage or current.

### 1.1.5 Element Types

The following are common types of circuit element.

- **Energy storage elements** store energy in electric (capacitors) or magnetic (inductors) fields.
- **Energy dissipative elements** dissipate energy from a circuit, typically as heat, such as in a resistor.
- **Energy source elements** provide external energy to the circuit (e.g. batteries).
- **Energy transducing elements** convert electronic energy to another form (e.g. motors convert electric to mechanical energy.)

### 1.1.6 Power

Power is the time rate of change of energy. Let us now define electric power.

#### Definition 1.4

The instantaneous electric power  $\mathcal{P}$  into a circuit element is defined as the product of the voltage  $v$  across and the current  $i$  through it at a given time  $t$ :

$$\mathcal{P}(t) = v(t)i(t).$$

Power typically goes into:

- heat (usually),
- mechanical work (motors),
- radiated energy (lamps, transmitters), or
- stored energy (batteries, capacitors).

#### Box 1.1

##### Terminological Note

"[D]on't call current 'amperage'; that's strictly bush-league. The same caution will apply to the term 'ohmage' ..."

—Horowitz & Hill, The Art of Electronics

### 1.1.7 Kirchhoff's Laws

Gustav Kirchhoff formulated two laws fundamental to circuit analysis.

Kirchhoff's current law (KCL) depends on the fact that charge is a conserved quantity. Therefore, the charge flowing *in* a node is equal to that flowing *out*, which implies KCL.

#### Definition 1.5

The current *in* a node is equal to the current *out*.

KCL implies that the sum of the current into a node must be zero. Assume, for instance, that  $k$  wires with currents  $i_j$  connect to form a node. Kirchhoff's current law states that

$$\sum_{j=1}^k i_j = 0.$$

It can be discovered empirically that elements connected in parallel have the same voltage across them. This doesn't mean they share the same current, but it does imply Kirchhoff's voltage law (KVL).

### Definition 1.6

The sum of the voltage drops around any closed loop is zero.<sup>a</sup>

a. A loop is a series of elements that begins and ends at the same node.

KVL implies that the voltage drops across elements that form a loop must be zero. Assume, for instance, that  $k$  elements with voltage drops  $v_j$  form a loop. KVL states that

$$\sum_{j=1}^k v_j = 0.$$

### 1.1.8 Ohm's Law

Much of electronics is about the relationship between a voltage and a corresponding current. Applying a voltage to a material typically induces a current through it. The functional relationship between  $v$  and  $i$  is of the utmost importance to the analysis and design of circuits.

The simplest relationship is known as Ohm's law, for which we will first need the concept of resistance.

### Definition 1.7

Let a circuit element have voltage  $v$  and current  $i$ . The resistance  $R$  is defined as the ratio

$$R = v/i$$

Now we are ready to define Ohm's law.

### Definition 1.8

Some materials such as conductors in certain environments exhibit approximately constant resistance.

This is pretty weak. However, it's still quite useful, as we'll see. With it we can assume, for certain elements and situations, that the resistance of the element is a static property and that the voltage and current are proportional. We call such elements **resistors**.

### 1.1.9 Combining Resistance

Resistors can be connected together in different topologies to form composite elements that exhibit "equivalent" resistances of their own.

$K$  resistors with resistances  $R_j$  connected in *series* have equivalent resistance  $R_e$  given by the expression

$$R_e = \sum_{j=1}^K R_j.$$

$K$  resistors with resistances  $R_j$  connected in *parallel* have equivalent resistance  $R_e$  given by the expression

$$R_e = 1 / \sum_{j=1}^K 1/R_j.$$

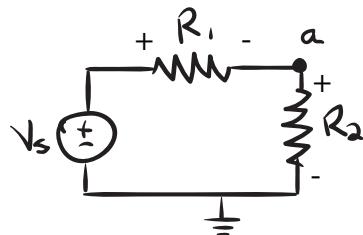
In the special case of two resistors with resistances  $R_1$  and  $R_2$ ,

$$\begin{aligned} R_e &= \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} \\ &= \frac{R_1 R_2}{R_1 + R_2}. \end{aligned}$$

#### Example 1.1

Answer the questions below about the circuit shown. Voltage across and current through a circuit element  $x$  are denoted  $v_x$  and  $i_x$ . Signs are defined on the diagram.

1. What does it mean if we refer to the voltage at node  $a$ ?
2. What is the current  $i_{R_2}$  through  $R_2$  at a given time  $t$  in terms of the power it is dissipating  $\mathcal{P}_{R_2}$  and the voltage across it  $v_{R_2}$ ?
3. If  $V_s(t) = 5$  V and  $v_{R_1} = 3$  V, what is  $v_{R_2}$ ?
4. What is the equivalent resistance of the resistors  $R_1$  and  $R_2$  combined as in the circuit?
5. If  $v_{R_1} = 3$  V and  $R_1 = 100 \Omega$ , what is  $i_{R_2}$ ?



1. We mean the voltage from  $a$  to ground.
2. From the definition of the instantaneous power,

$$i_{R_2}(t) = \mathcal{P}_{R_2}(t) / v_{R_2}(t).$$

3. From KVL,

$$\begin{aligned} v_{R_2} &= V_s - v_{R_1} \\ &= 5 - 3 \\ &= 2 \text{ V.} \end{aligned}$$

4. In the circuit,  $R_1$  and  $R_2$  are in *series*, so they can be combined as

$$R_e = R_1 + R_2.$$

5. We can write down Ohm's law for  $R_1$ :

$$v_{R_1} = i_{R_1} R_1.$$

So we know

$$\begin{aligned} i_{R_1} &= v_{R_1} / R_1 \\ &= 30 \text{ mA.} \end{aligned}$$

From KCL,  $i_{R_2} = i_{R_1}$ . Therefore,

$$i_{R_2} = 30 \text{ mA.}$$

### Box 1.2 Further Reading

- Horowitz and Hill (2015; chapter 1), Required reading for the serious student of electronics

## 1.2 Voltage Dividers



In chapter 2 we'll learn about how to approach circuit analysis in a systematic way. For now, we'll limp along unsystematically with our toolbelt of concepts and equations in order to introduce some more circuit elements, concepts, and theorems. But we can't resist just a bit of circuit analysis now.

The **voltage divider** is a ubiquitous and useful circuit. In a sense, it's less of a circuit and more of a concept. For resistors, that concept can be stated as the following.

The voltage across resistors in series is divided among the resistors.

An immediately useful result is that we can "divide voltage" into any smaller voltage we like by putting in a couple resistors.

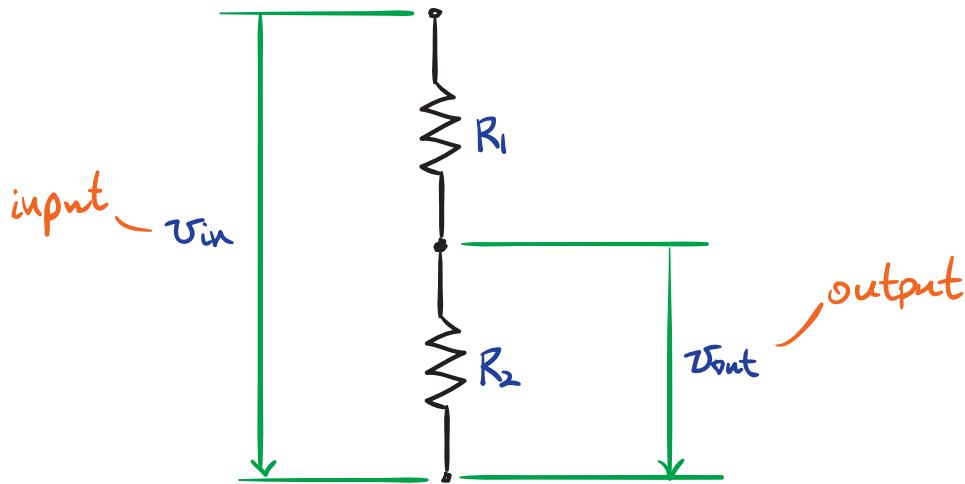


Figure 1.1. a simple voltage divider circuit.

In order to show *how* the voltage divider "divides up" the voltage, we must do some basic circuit analysis. Consider the circuit in figure 1.1. The input voltage  $v_{in}$  is divided into  $v_{R_1}$  and  $v_{R_2} = v_{out}$ . We want to know  $v_{out}$  as a function of  $v_{in}$  and parameters  $R_1$  and  $R_2$ . Let's write down the equations we know from the laws of Kirchhoff and Ohm:

$$v_{R_1} = i_{R_1} R_1,$$

$$v_{R_2} = i_{R_2} R_2,$$

$$v_{in} = v_{R_1} + v_{R_2}, \text{ and}$$

$$i_{R_1} = i_{R_2}.$$

We've already established that  $v_{\text{out}} = v_{R_2}$ , so we can solve for  $v_{R_2}$  in (\*). We want to eliminate the three "unknown" variables  $v_{R_1}$ ,  $i_{R_1}$ , and  $i_{R_2}$ , so it is good that we have four equations.<sup>2</sup> We begin with (\*b) and proceed by substitution of the others of (\*):

$$\begin{aligned}
 v_{R_2} &= i_{R_2} R_2 \\
 &= i_{R_1} R_2 \\
 &= \frac{R_2}{R_1} v_{R_1} \\
 &= \frac{R_2}{R_1} (v_{\text{in}} - v_{R_2}) \Rightarrow \\
 v_{R_2} + \frac{R_2}{R_1} v_{R_2} &= \frac{R_2}{R_1} v_{\text{in}} \Rightarrow \\
 v_{R_2} &= \frac{R_2 / R_1}{1 + R_2 / R_1} v_{\text{in}} \Rightarrow \\
 &= \frac{R_2}{R_1 + R_2} v_{\text{in}}.
 \end{aligned}$$

Nice! So we can now write the input-output relationship for a two-resistor voltage divider.

### Definition 1.9

For a voltage  $v_{\text{in}}$  split across two resistors  $R_1$  and  $R_2$  in series, the voltage divider equation is

$$v_{R_2} = \frac{R_2}{R_1 + R_2} v_{\text{in}}$$

for  $R_2$  or

$$v_{R_1} = \frac{R_1}{R_1 + R_2} v_{\text{in}}$$

for  $R_1$ .

So the voltage divider had the effect of dividing the input voltage into a fraction governed by the relationship between the relative resistances of the two resistors. This fraction takes values in the interval  $[0, 1]$ . Now, whenever we see the voltage divider circuit, we can just remember this easy formula!

Similarly, for  $n$  resistors in series, it can be shown that the voltage divider relationship is as follows.

2. Alternatively, we could solve for all four unknown variables with our four equations.

**Definition 1.10**

For a voltage  $v_{\text{in}}$  split across  $n$  resistors  $R_1, R_2, \dots, R_n$  in series, the voltage divider equation is

$$v_{R_m} = \left( R_m \left/ \sum_{k=1}^n R_k \right. \right) v_{\text{in}}$$

for resistor  $R_m$ .

**1.3 Sources**

Sources (i.e., supplies) supply power to a circuit. There are two primary types: *voltage sources* and *current sources*.

**1.3.1 Ideal Voltage Sources**

An ideal voltage source provides exactly the voltage a user specifies, independent of the circuit to which it is connected. All it must do in order to achieve this is to supply whatever current necessary. Let's unpack this with a simple example.

**Example 1.2**

In the circuit shown, determine how much current and power the ideal voltage source  $V_s$  must provide in order to maintain voltage if  $R \rightarrow \infty$  and if  $R \rightarrow 0$ .

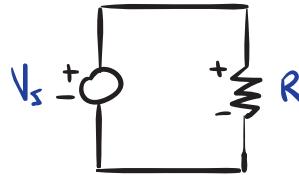


Figure 1.2

Since the voltage across the resistor is known to be equal to  $V_s$ , Ohm's law tells us that

$$i_R = V_s/R.$$

Of course, power dissipated by the resistor is

$$\mathcal{P}_R = i_R v_R$$

$$= V_s^2/R.$$

Taking the “open circuit” limit,

$$\begin{aligned} i_{R \rightarrow \infty} &= \lim_{R \rightarrow \infty} V_s / R \\ &= 0 \text{ A} \end{aligned}$$

$$\begin{aligned} \mathcal{P}_{R \rightarrow \infty} &= \lim_{R \rightarrow \infty} V_s^2 / R \\ &= 0 \text{ W.} \end{aligned}$$

Taking the “short circuit” limit,

$$\begin{aligned} i_{R \rightarrow 0} &= \lim_{R \rightarrow 0} V_s / R \\ &\rightarrow \infty \text{ A} \end{aligned}$$

$$\begin{aligned} \mathcal{P}_{R \rightarrow 0} &= \lim_{R \rightarrow 0} V_s^2 / R \\ &\rightarrow \infty \text{ W.} \end{aligned}$$

### 1.3.2 Ideal Current Sources

An ideal current source provides exactly the current a user specifies, independent of the circuit to which it is connected. All it must do in order to achieve this is to supply whatever voltage necessary. Let’s unpack this with a simple example.

#### Example 1.3

In the circuit shown, determine how much voltage and power the ideal current source  $I_s$  must provide in order to maintain voltage if  $R \rightarrow 0$  and if  $R \rightarrow \infty$ .

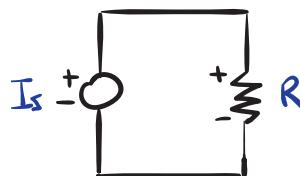


Figure 1.3

Since the current through the resistor is known to be equal to  $I_s$ , Ohm’s law tells us that

$$v_R = I_s R.$$

Of course, power dissipated by the resistor is

$$\begin{aligned}\mathcal{P}_R &= i_R v_R \\ &= I_s^2 R.\end{aligned}$$

Taking the “short circuit” limit,

$$\begin{aligned}v_{R \rightarrow 0} &= \lim_{R \rightarrow 0} I_s R \\ &\rightarrow 0 \text{ A}\end{aligned}$$

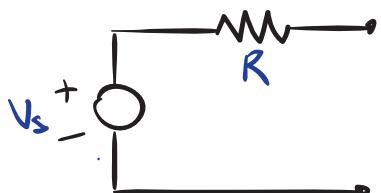
$$\begin{aligned}\mathcal{P}_{R \rightarrow 0} &= \lim_{R \rightarrow 0} I_s^2 R \\ &\rightarrow 0 \text{ W}.\end{aligned}$$

Taking the “open circuit” limit,

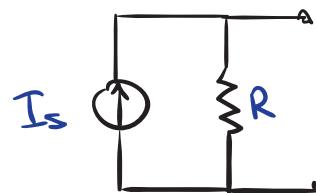
$$\begin{aligned}v_{R \rightarrow \infty} &= \lim_{R \rightarrow \infty} I_s R \\ &= \infty \text{ A} \\ \mathcal{P}_{R \rightarrow \infty} &= \lim_{R \rightarrow \infty} I_s^2 R \\ &= \infty \text{ W}.\end{aligned}$$

### 1.3.3 Modeling Real Sources

No real source can produce infinite power. Some have feedback that controls the output within some finite power range. These types of sources can be approximated as ideal when operating within their specifications. Many voltage sources (e.g. batteries) do not have internal feedback controlling the voltage. When these sources are “loaded” (delivering power) they cannot maintain their nominal output, be that voltage or current. We model these types of sources as ideal sources in series or parallel with a resistor, as illustrated in figure 1.4.



(a) Real voltage source model



(b) Real current source model.

Figure 1.4. Models for power-limited “real” sources.

Most manufacturers specify the nominal resistance of a source as the “output resistance.” A typical value is  $50\ \Omega$ .

## 1.4 Thévenin’s and Norton’s Theorems

Thévenin’s and Norton’s theorems yield ways to simplify our models of circuits.



### 1.4.1 Thévenin’s Theorem

The following remarkable theorem has been proven.

#### Theorem 1.11

Thévenin’s theorem: Given a linear network of voltage sources, current sources, and resistors, the behavior at the network’s output terminals can be reproduced exactly by a single *voltage source  $V_e$  in series with a resistor  $R_e$* .

The equivalent circuit has two quantities to determine:  $V_e$  and  $R_e$ .

**1.4.1.1 Determining  $R_e$**  The **equivalent resistance  $R_e$**  of a circuit is the resistance between the output terminals with all inputs set to zero. Setting a voltage source to zero means the voltage on both its terminals are equal, which is equivalent to treating it as a short or wire. Setting a current source to zero means the current through it is zero, which is equivalent to treating it as an open circuit.

**1.4.1.2 Determining  $V_e$**  The **equivalent voltage source  $V_e$**  is the voltage at the output terminals of the circuit when they are left open (disconnected from a load). Determining this value typically requires some circuit analysis with the laws of Ohm and Kirchhoff.

### 1.4.2 Norton’s Theorem

Similarly, the following remarkable theorem has been proven.

#### Theorem 1.12

Norton’s theorem: Given a linear network of voltage sources, current sources, and resistors, the behavior at the network’s output terminals can be reproduced exactly by a single *current source  $I_e$  in parallel with a resistor  $R_e$* .

The equivalent circuit has two quantities to determine:  $I_e$  and  $R_e$ . The equivalent resistance  $R_e$  is identical to that of Thévenin’s theorem, which leaves the equivalent current source  $I_e$  to be determined.

**1.4.2.1 Determining  $I_e$**  The **equivalent current source**  $I_e$  is the current through the output terminals of the circuit when they are shorted (connected by a wire). Determining this value typically requires some circuit analysis with the laws of Ohm and Kirchhoff.

### 1.4.3 Converting Between Thévenin and Norton Equivalents

There is an equivalence between the two equivalent circuit models that allows one to convert from one to another with ease. The equivalent resistance  $R_e$  is identical in each and provides the following equation for converting between the two representations: converting between Thévenin and Norton equivalents

#### Example 1.4

For the circuit shown, find a Thévenin and a Norton equivalent circuits.

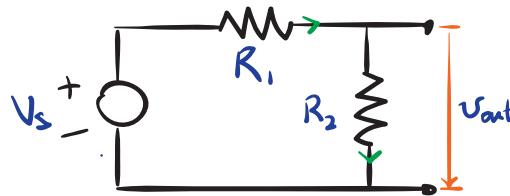


Figure 1.5

The Thévenin equivalent is shown. Now to find  $R_e$  and  $V_e$ . Setting  $V_s = 0$ ,  $R_1$  and  $R_2$  are in parallel, combining to give

$$R_e = \frac{R_1 R_2}{R_1 + R_2}.$$

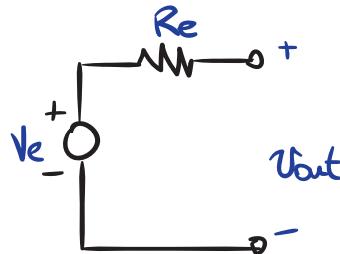


Figure 1.6

Now to find  $v_{\text{out}}$ . It's a voltage divider:

$$V_e = v_{\text{out}} = \frac{R_2}{R_1 + R_2} V_s.$$

The Norton equivalent is shown. We know  $R_e$  from the Thévenin equivalent, which also yields

$$I_e = V_e / R_e.$$

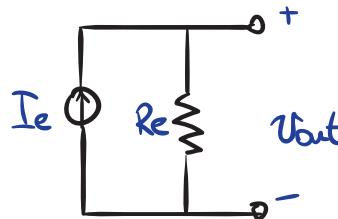


Figure 1.7

## 1.5 Output and Input Resistance and Circuit Loading



When considering a circuit from the perspective of two terminals—either as *input* or *output*—it is often characterized as having a Thévenin/Norton **equivalent resistance** and, if it is considered as an output, as having an equivalent (Thévenin or Norton) source.

If the terminals are considered to be an *output*, its **output resistance** is just the Thévenin/Norton equivalent resistance. Other names for this output resistance are *source* or *internal resistance*.<sup>3</sup> Figure 1.8 illustrates this model.

3. Sometimes, instead of *resistance*, the term *impedance* is substituted. In these situations, there is no difference in meaning.

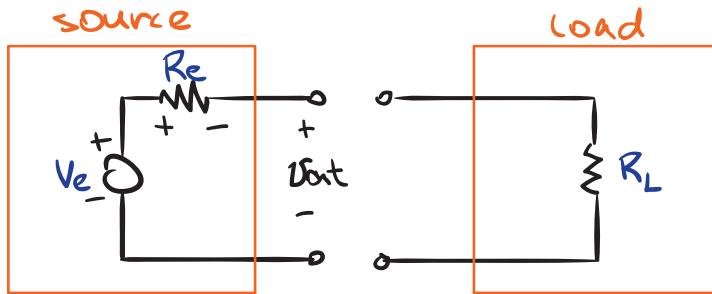


Figure 1.8. Source with Thévenin equivalent source voltage  $V_e$  and output/internal resistance  $R_e$  and a load with input resistance  $R_L$ .

If the terminals are considered to be an *input*, its **input resistance** is the is the Thévenin/Norton equivalent resistance of the circuit. Another term for this input resistance is the *load resistance*.

### 1.5.1 Loading the Source

**Loading a source** means to connect another circuit to it that draws power. Let's explore what happens when we connect the load to the source for the circuit in figure 1.8.

Before connecting, the source output voltage is

$$\begin{aligned}
 v_{out} &= V_e - v_{R_e} \\
 &= V_e - \overset{0}{\uparrow}_{R_e} R_e \\
 &= V_e.
 \end{aligned}$$

This is equivalent to connecting a load with an infinite resistance. After connecting, we have a voltage divider, so

$$\begin{aligned}
 v_{out} &= \frac{R_L}{R_e + R_L} V_e \\
 &= \frac{1}{1 + R_e/R_L} V_e.
 \end{aligned}$$

So, as  $R_e/R_L \rightarrow 0$ ,  $v_{out} \rightarrow V_e$ . Also, as  $R_e/R_L \rightarrow \infty$ ,  $v_{out} \rightarrow 0$ .

So, relatively small output resistance and large input resistance yield a “loaded” voltage nearer nominal. Some sources are labeled with nominal values assuming no

load and others assuming a **matching load**<sup>4</sup>—a load equal to the output impedance. For this reason, it is best to measure the actual output of any source.

## 1.6 Capacitors

Capacitors have two terminal and are composed of two conductive surfaces separated by some distance. One surface has charge  $q$  and the other  $-q$ . A capacitor stores energy in an *electric field* between the surfaces.

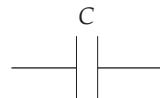


Let a capacitor with voltage  $v$  across it and charge  $q$  be characterized by the parameter **capacitance**  $C$ , where the constitutive equation is

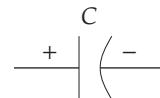
$$q = Cv.$$

The capacitance has derived SI unit **farad** (F), where  $F = A \cdot s/V$ . A farad is actually quite a lot of capacitance. Most capacitors have capacitances best represented in  $\mu F$ ,  $nF$ , and  $pF$ .

The time-derivative of this equation yields the  $v$ - $i$  relationship (what we call the “elemental equation”) for capacitors. capacitor elemental equation A time-derivative! This is new. Resistors have only algebraic  $i$ - $v$  relationships, so circuits with only sources and resistors can be described by *algebraic* relationships. The dynamics of circuits with capacitors are described with *differential equations*.



(a) bipolar capacitor.



(b) polarized capacitor

Figure 1.9. capacitor circuit diagram symbols.

Capacitors allow us to build many new types of circuits: filtering, energy storage, resonant, blocking (blocks dc-component), and bypassing (draws ac-component to ground).

Capacitors come in a number of varieties, with those with the largest capacity (and least expensive) being **electrolytic** and most common being **ceramic**. There are two functional varieties of capacitors: **bipolar** and **polarized**, with circuit diagram symbols shown in figure 1.9. Polarized capacitors can have voltage drop across in only one direction, from **anode** (+) to **cathode** (−)—otherwise they are damaged or may **explode**. Electrolytic capacitors are polarized and ceramic capacitors are bipolar.

4. A matching load can be shown to have maximum power transfer.

So what if you need a high-capacitance bipolar capacitor? Here's a trick: place identical high-capacity polarized capacitors **cathode-to-cathode**. What results is effectively a bipolar capacitor with capacitance *half* that of one of the polarized capacitors.

## 1.7 Inductors

A **pure inductor** is defined as an element in which **flux linkage**  $\lambda$ —the time integral of the voltage—across the inductor is a monotonic function  $\mathcal{F}$  of the current  $i$ ; i.e. the pure constitutive equation is

$$\lambda = \mathcal{F}(i).$$



An **ideal inductor** is such that this monotonic function is linear, with slope called the **inductance**  $L$ ; i.e. the ideal constitutive equation is

$$\lambda = Li$$

The units of inductance are the SI derived unit **henry (H)**. Most inductors have inductance best represented in mH or  $\mu$ H.

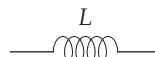


Figure 1.10. Inductor circuit diagram symbol.

The elemental equation for an inductor is found by taking the time-derivative of the constitutive equation. **inductor elemental equation**

Inductors store energy in a *magnetic field*. It is important to notice how inductors are, in a sense, the *opposite* of capacitors. A capacitor's current is proportional to the time rate of change of its voltage. An inductor's voltage is proportional to the time rate of change of its current.

Inductors are usually made of wire coiled into a number of turns. The geometry of the coil determines its inductance  $L$ .

Often, a **core** material—such as iron and ferrite—is included by wrapping the wire around the core. This increases the inductance  $L$ .

Inductors are used extensively in radio-frequency (rf) circuits, with which we won't discuss in this text. However, they play important roles in ac-dc conversion, filtering, and transformers—all of which we will consider extensively.

The circuit diagram for an inductor is shown in figure 1.10.

---

## 1.8 Problems

**Problem 1.1**

- Let two resistors with resistances  $1\text{ k}\Omega$  and  $2\text{ k}\Omega$  be connected in series. What is their combined effective resistance?
- Let two resistors  $R_1$  and  $R_2$  be connected in series. Prove that their combined effective resistance is greater than that of either resistor, individually. Use KVL, KCL, and Ohm's Law.
- Let two resistors with resistances  $1\text{ k}\Omega$  and  $2\text{ k}\Omega$  be connected in parallel. What is their combined effective resistance?
- Let any two resistors  $R_1$  and  $R_2$  be connected in parallel. Prove that their combined effective resistance is less than that of either resistor, individually. Use KVL, KCL, and Ohm's Law.

**Problem 1.2** Beginning with the definition of electrical power and the elemental equation of an ideal resistor, find

- an expression for the power dissipated by a resistor in terms of voltage  $v_R$  and resistance  $R$ , only; and
- an expression for the power dissipated by a resistor in terms of current  $i_R$  and resistance  $R$ , only.

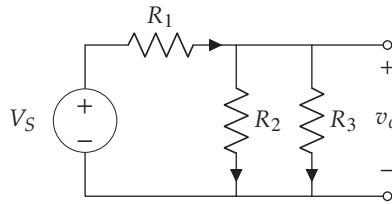
**Problem 1.3** An unregulated function generator has a  $50\text{ }\Omega$  output resistance. The front panel displays a nominal voltage amplitude of  $10\text{ V}$ , which assumes a matching load of  $50\text{ }\Omega$ . However, the output is *not* connected to this nominal matching load. Instead, it is connected to an oscilloscope with high input resistance—let's say it's infinite. Respond to the following questions and imperatives about this situation.

- Draw a circuit diagram.
- Using the given information about the “nominal” voltage amplitude, determine what the ideal source voltage amplitude  $V_s$  should be in your circuit diagram/function generator model.
- Solve for the actual voltage amplitude  $v_a$  at the oscilloscope if the front panel says  $5\text{ V}$  amplitude.

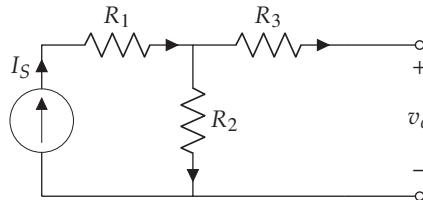
**Problem 1.4**  **CORPORATIONISM** Consider two signals with voltage ratios expressed in decibels as follows. What are the corresponding power and voltage amplitude ratios?<sup>5</sup>

- a. 0 dB
- b. 3 dB
- c. 10 dB
- d. 20 dB

**Problem 1.5**  **PSEUDOSCARUS** For the circuit diagram below with voltage source  $V_S$  and output voltage  $v_o$ , (a) construct a Thévenin equivalent circuit. Be sure to specify the equivalent source  $V_e$  and resistance  $R_e$ . Let  $R_1 = R_2 = 1 \text{ k}\Omega$  and  $R_3 = 2 \text{ k}\Omega$ . (b) Convert the Thévenin equivalent circuit from (a) to a Norton equivalent.

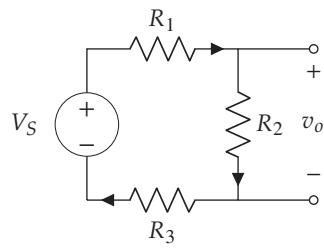


**Problem 1.6**  **BANANA** For the circuit diagram below with current source  $I_S$  and output voltage  $v_o$ , (a) construct a Norton equivalent circuit. Be sure to specify the equivalent source  $I_e$  and resistance  $R_e$ . Let  $R_1 = R_2 = 1 \text{ k}\Omega$  and  $R_3 = 2 \text{ k}\Omega$ . (b) Convert the Norton equivalent circuit from (a) to a Thévenin equivalent.



**Problem 1.7**  **DOORBELL** For the circuit diagram below with voltage source  $V_S$  and output voltage  $v_o$ , (a) construct a Norton equivalent circuit. Be sure to specify the equivalent source  $I_e$  and resistance  $R_e$ . Let  $R_1 = 1 \text{ k}\Omega$ ,  $R_2 = 2 \text{ k}\Omega$ , and  $R_3 = 3 \text{ k}\Omega$ . (b) Convert the Norton equivalent circuit from (a) to a Thévenin equivalent.

5. This exercise was inspired by Horowitz and Hill (2015).



# 2 Circuit Analysis



In this chapter, we consider the techniques of analog circuit analysis. We use a time-domain approach in which the dynamics of circuits are expressed as ordinary differential equations (ODEs).

## 2.1 Sign Convention

We use the **passive sign convention** of electrical engineering, defined below and illustrated in figure 2.1.



### Definition 2.1

Power flowing *in* to a component is considered to be *positive* and power flowing *out* of a component is considered *negative*.

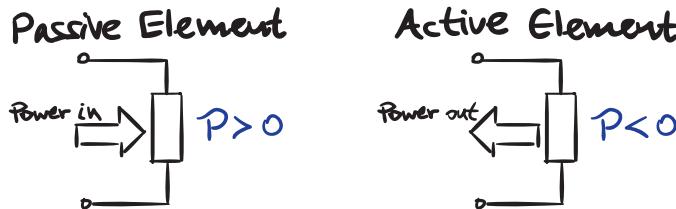


Figure 2.1. passive sign convention in terms of power  $\mathcal{P}$ .

Because power  $\mathcal{P} = vi$ , this implies the current and voltage signs are prescribed by the convention. For **passive elements**, the electrical potential must drop in the direction of positive current flow. This means the assumed direction of voltage drop across a passive element must be the same as that of the current flow. For **active elements**, which supply power to the circuit, the converse is true: the voltage drop

and current flow must be in opposite directions. figure 2.2 illustrates the possible configurations.

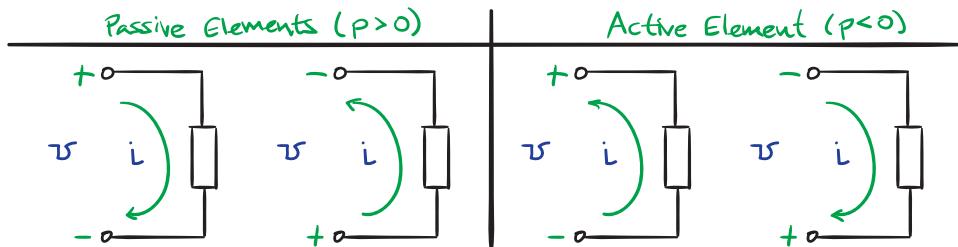


Figure 2.2. passive sign convention in terms of voltage  $v$  and current  $i$ .

When analyzing a circuit, for each passive element, draw an arrow beside it pointing in the direction of assumed current flow and voltage drop. Try it out on figure 2.3.

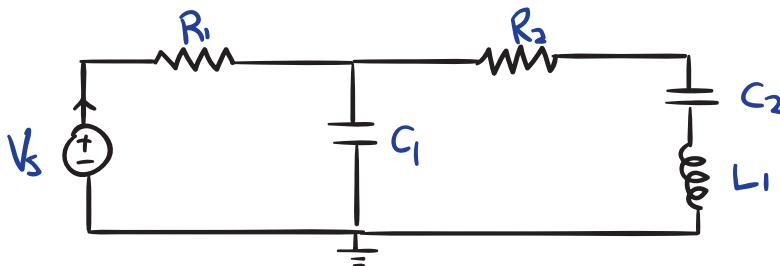


Figure 2.3. an illustration of the passive sign convention on a circuit.

The purpose of a sign convention is to help us **interpret** the signs of our results. For instance, if, at a given instant, a capacitor has voltage  $v_C = 3$  V and current  $i_C = -2$  A, we compute  $\mathcal{P}_C = -6$  W and we know 6 W of power is flowing *from* the capacitor into the circuit.

For passive elements, there is no preferred direction of “assumed” voltage drop and current flow. If a voltage or current value discovered by performing a circuit analysis is positive, this means the “assumed” and “actual” directions are the same. For a negative value, the directions are opposite.

For active elements, *we don't get to choose* the direction. The physical situation prescribes it. For instance, if a +12 V positive terminal of a battery is connected

to a certain terminal in a circuit, the voltage of that terminal should be assigned a positive voltage of +12 V.

## 2.2 Methodology for Analyzing Circuits

We have all the tools we need to do some pretty badass circuit analysis.



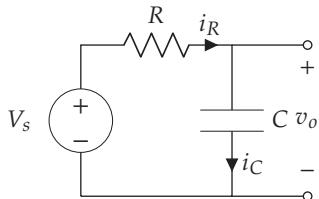
Later we'll learn a more systematic method for analyzing the dynamics of a circuit, but for now we can use broad strokes to get the idea. It will work most of the time, but occasionally you may need to write some extra KCL or KVL equations or use a more advanced algebraic technique.

Let  $n$  be the number of passive circuit elements in a circuit, which gives  $2n$  ( $v$  and  $i$  for each element) unknowns. The method is this.

1. Draw a *circuit diagram*.
2. Label the circuit diagram with the *sign assignment* by labeling each element with the "assumed" direction of current flow.
3. Write the *elemental equation* for each circuit element (e.g. Ohm's law).
4. For every node not connected to a voltage source, write Kirchhoff's current law (KCL).
5. For each loop not containing a current source, write Kirchhoff's voltage law (KVL).
6. You probably have a linear system of  $2n$  algebraic and first-order, ordinary differential equations (and  $2n$  unknowns) to be solved simultaneously.
  1. Eliminate  $n$  (half) of the unknowns by substitution into the elemental equations.
  2. Try substitution or elimination to get down to only those variables with time derivatives and inputs. If this doesn't work, use a linear algebra technique.
  3. Solve the remaining set of first-order, linear ordinary differential equations. This can be done either directly or by turning it into a single higher-order differential equation and then solving.

### Example 2.1

In the RC circuit shown, let  $V_s(t) = 12$  V. If  $v_C(t)|_{t=0} = 0$ , what is  $v_o(t)$  for  $t \geq 0$ ?



1. The circuit diagram is given.
2. The signs are given.
3. The  $n$  elemental equations are as follows.

$$\begin{array}{ll} C & \frac{dv_C}{dt} = \frac{1}{C} i_C \\ R & v_R = i_R R \end{array}$$

4. There is one node not connected to the voltage source, for which the KCL equation is

$$i_C = i_R.$$

5. There is one loop which has KVL equation is

$$v_R + v_C - V_s = 0.$$

6. Solve.

1. Eliminate  $i_C$  and  $v_R$  using KCL and KVL to yield the following.

$$\begin{aligned} C \frac{dv_C}{dt} &= \frac{1}{C} i_R \\ R i_R &= \frac{1}{R} (V_s - v_C) \end{aligned}$$

2. Substituting the  $R$  equation into the  $C$  equation, we eliminate  $i_R$ :

$$\frac{dv_C}{dt} = \frac{1}{RC} (V_s - v_C).$$

3. Rearranging,

$$RC \frac{dv_C}{dt} + v_C = V_s.$$

Let's solve the ODE with initial condition  $v_C(0) = 0$ .

1. Homogeneous solution. The characteristic equation is

$$RC\lambda + 1 = 0 \Rightarrow \lambda = -\frac{1}{RC}.$$

Therefore,  $v_{Ch}(t) = k_1 e^{-t/(RC)}$ .

2. Particular solution. The form of the input  $V_s(t)$  invites us to assume  $v_{Cp}(t) = k_2$ . Substituting into the ODE,

$$RC(0) + k_2 = 12 \text{ V} \Rightarrow k_2 = 12 \text{ V}.$$

3. Total solution:  $v_C(t) = v_{Ch}(t) + v_{Cp}(t)$ .

4. Specific solution:

$$v_C(0) = 0 \Rightarrow k_1 + k_2 = 0 \Rightarrow k_1 = -k_2 = -12 \text{ V}.$$

Since,  $v_o(t) = v_C(t)$ ,

$$v_o(t) = 12(1 - e^{-t/(RC)}).$$

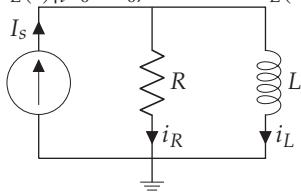
## 2.3 A Sinusoidal Input Example

Notice that we have yet to talk about **alternating current (ac) circuit analysis** or **direct current (dc) circuit analysis**. In fact, these ambiguous terms can mean a few different things. Approximately, an ac circuit analysis is one for which the input is sinusoidal and a dc circuit analysis is one for which the input is a constant. This ignores **transient response** (early response when the initial-condition response dominates) versus **steady-state response** (later response when the initial-condition response has decayed) considerations. We'll consider this more in (lec:transient\_steady).

We have remained general enough to be able to handle sinusoidal and constant sources in both transient and steady-state response. ([ex:rc\\_circuit\\_analysis\\_01](#)) features a circuit with a constant voltage source and a capacitor. Now we consider circuit with a sinusoidal current source and an inductor because why change only one thing when you could change more?

### Example 2.2

Given the RL circuit shown, current input  $I_s(t) = A \sin \omega t$ , and initial condition  $i_L(t)|_{t=0} = i_0$ , what are  $i_L(t)$  and  $v_L(t)$  for  $t \geq 0$ ?



1. The circuit diagram is given.
2. The signs are given.
3. The  $n$  elemental equations are as follows.

$$L \quad \frac{di_L}{dt} = \frac{1}{L} v_L \quad (2.1)$$

$$R \quad v_R = i_R R \quad (2.2)$$

4. There are actually two nodes not connected to the voltage source, but they give the same KCL equation

$$i_R = I_s - i_L.$$

5. There is one loop that doesn't have a current source in it, for which the KVL equation is

$$v_L = v_R.$$

6. Solve.

1. Eliminate  $v_L$  and  $i_R$  using KCL and KVL to yield the following.

$$L \frac{di_L}{dt} = \frac{1}{L} v_R \quad (2.3)$$

$$R v_R = (I_s - i_L)R \quad (2.4)$$

2. Substituting the  $R$  equation into the  $L$  equation, we eliminate  $v_R$  to obtain

$$\frac{di_L}{dt} = \frac{R}{L} (I_s - i_L).$$

3. Rearranging and letting  $\tau = L/R$ ,

$$\tau \cdot \frac{di_L}{dt} + i_L = I_s.$$

Let's solve the ODE with initial condition  $i_L(0) = i_0$ .

1. Homogeneous solution. The characteristic equation is

$$\tau \lambda + 1 = 0 \Rightarrow \lambda = -\frac{1}{\tau}.$$

Therefore,  $i_{Lh}(t) = \kappa_1 e^{-t/\tau}$ .

2. Particular solution. The form of the input  $I_s(t)$  invites us to assume  $i_{Lp}(t) = k_1 \sin \omega t + k_2 \cos \omega t$ . Substituting into the ODE,

$$\begin{aligned} & \tau \omega (k_1 \cos \omega t - k_2 \sin \omega t) + \\ & (k_1 \sin \omega t + k_2 \cos \omega t) = A \sin \omega t \Rightarrow \\ & \tau \omega k_1 + k_2 = 0 \quad \text{and} \quad -\tau \omega k_2 + k_1 = A \Rightarrow \\ & k_2 = -\tau \omega k_1 \quad \text{and} \quad (\tau \omega)^2 k_1 + k_1 = A \Rightarrow \\ & k_1 = \frac{A}{(\tau \omega)^2 + 1} \quad \text{and} \quad k_2 = -\frac{A \tau \omega}{(\tau \omega)^2 + 1} \end{aligned}$$

Success! So we have

$$i_{Lp}(t) = \frac{A}{(\tau \omega)^2 + 1} (\sin \omega t - \tau \omega \cos \omega t).$$

3. Total solution:

$$\begin{aligned} i_L(t) &= i_{Lh}(t) + i_{Lp}(t) \\ &= \kappa_1 e^{-t/\tau} + \frac{A}{(\tau \omega)^2 + 1} (\sin \omega t - \tau \omega \cos \omega t). \end{aligned}$$

4. Specific solution:

$$\begin{aligned} i_L(0) &= i_0 \quad \Rightarrow \\ \kappa_1 - \frac{A\tau\omega}{(\tau\omega)^2 + 1} &= i_0 \quad \Rightarrow \\ \kappa_1 &= i_0 + \frac{A\tau\omega}{(\tau\omega)^2 + 1}. \end{aligned}$$

Therefore,

$$\begin{aligned} i_L(t) &= \left( i_0 + \frac{A\tau\omega}{(\tau\omega)^2 + 1} \right) e^{-t/\tau} + \\ &\quad \frac{A}{(\tau\omega)^2 + 1} (\sin \omega t - \tau\omega \cos \omega t). \end{aligned}$$

Now, a better way to write this is as a single sinusoid. The two-to-one formulas can be applied to obtain

$$\begin{aligned} i_L(t) &= \left( i_0 + \frac{A\tau\omega}{(\tau\omega)^2 + 1} \right) e^{-t/\tau} + \\ &\quad \frac{A}{\sqrt{(\tau\omega)^2 + 1}} \sin(\omega t - \arctan \tau\omega). \end{aligned}$$

We can see clearly that there is a sinusoidal component and a decaying exponential component.

7. Finally, we can find  $v_L(t)$  from the inductor elemental equation:

$$\begin{aligned} v_L(t) &= L \frac{di_L}{dt} \\ &= -\frac{L}{\tau} \left( i_0 + \frac{A\tau\omega}{(\tau\omega)^2 + 1} \right) e^{-t/\tau} + \\ &\quad \frac{AL\omega}{\sqrt{(\tau\omega)^2 + 1}} \cos(\omega t - \arctan \tau\omega). \end{aligned}$$

8. See section 2.4 for plots.

## 2.4 Transient and Steady-State Response

Let's consider them response of the circuit in example 2.2. We found that the inductor had current and voltage responses

$$i_L(t) = \left( i_0 + \frac{A\tau\omega}{(\tau\omega)^2 + 1} \right) e^{-t/\tau} + \frac{A}{\sqrt{(\tau\omega)^2 + 1}} \sin(\omega t - \arctan \tau\omega)$$

and

$$v_L(t) = -\frac{L}{\tau} \left( i_0 + \frac{A\tau\omega}{(\tau\omega)^2 + 1} \right) e^{-t/\tau} + \frac{AL\omega}{\sqrt{(\tau\omega)^2 + 1}} \cos(\omega t - \arctan \tau\omega).$$

Note that the top line of each of these equations decays exponentially to zero. The response while this term dominates is the *transient response* and the response thereafter is the *steady-state response*.

In  $6\tau$  (six time constants) the exponential term has decayed to less than 1 %, so we often assume the other term will be dominating by that point.

We will plot  $i_L(t)$  and  $v_L(t)$  from above to illustrate transient and steady-state response.

Let's use Python to plot the response. We begin by loading two packages as follows:

```
import numpy as np  # For numerical calculations
import matplotlib.pyplot as plt  # For plotting
```

Plots cannot be created without some definition of parameters. Let us define them as follows:

```
R = 1  # (Ohms) resistance
L = 1e-3  # (H) inductance
i_0 = 10  # (A) initial current in inductor
A = 10  # (A) sinusoidal input amplitude
omega = 5e3  # (rad/s) sinusoidal input angular frequency
tau = L/R  # (s) time constant
```

The current and voltage can be defined as follows:



```

def i_L(t):
    return (i_0+A*tau*omega/((tau*omega)^2+1))*exp(-t/tau) + \
        (A/sqrt((tau*omega)^2+1)* \
        sin(omega*t - arctan2(tau*omega,1)))
def v_L(t):
    return -L/tau*(i_0+A*tau*omega/ \
        ((tau*omega)^2+1))*exp(-t/tau) + \
        (A*L*omega/sqrt((tau*omega)^2+1)* \
        cos(omega*t - arctan2(tau*omega,1)))

```

Now we turn to defining simulation parameters.

```

N = 201 # Number of points to plot
t_min = 0 # Minimum time
t_max = 8 * tau # Maximum time
t_s = np.linspace(t_min, t_max, N) # Array of time values

```

Now to create numerical arrays to plot.

```

i_Ls = [] # initializing sampled array
v_Ls = [] # initializing sampled array
for i in range(0, N):
    i_Ls.append(i_L(t_s[i])) # build array of values
    v_Ls.append(v_L(t_s[i])) # build array of values

```

We use the package `matplotlib` to plot the inductor voltage and current.

```

fig = plt.figure()
ax = plt.subplot(111)
ax.plot(t_s, i_Ls, 'b-', linewidth=2, label='$i_L(t)$') # plot
ax.plot(t_s, v_Ls, 'r-', linewidth=2, label='$v_L(t)$') # plot
ax.legend(loc='center left', bbox_to_anchor=(1, 0.5))
ax.set_xlabel('time (s)')
plt.show() # Display

```

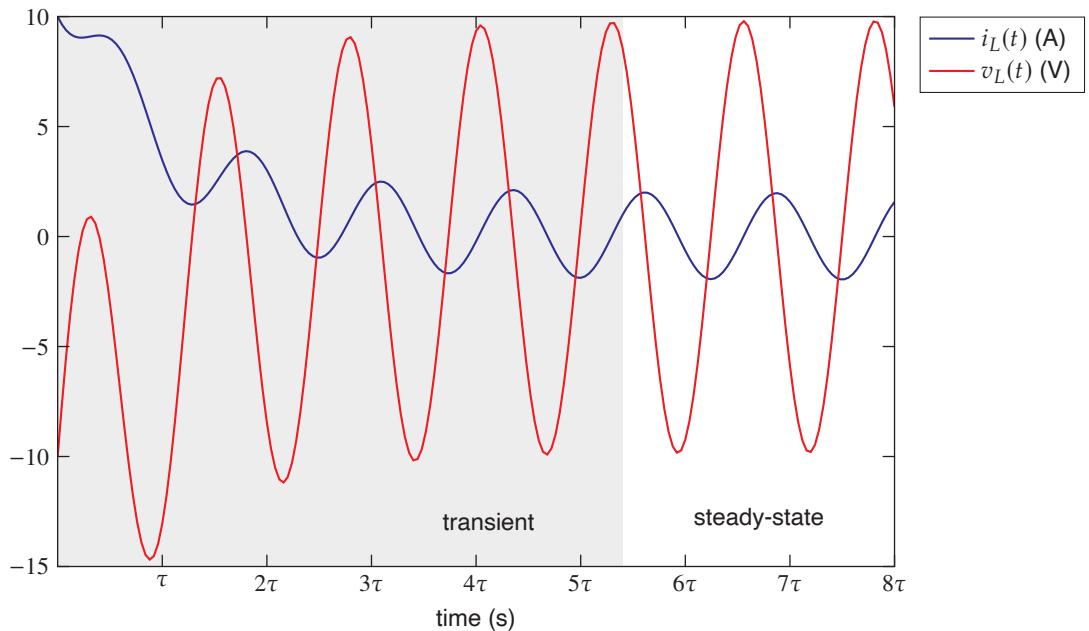


Figure 2.4. Current  $i_L$  and voltage  $v_L$  of the inductor for transient and steady-state response. Note that the transition is not precisely defined.

Figure 2.4 shows that in around six time constants, as is typical, the responses settle in to steady oscillations. Note that the steady-state is not necessarily *static*, but can also be oscillatory, as in this case. In fact, every linear dynamic system driven by a sinusoid will have a sinusoidal steady-state response, as we will explore further in the coming lectures.

Often the term *ac circuit analysis* is used to refer to circuits with sinusoidal sources *in steady-state*. In many circuits, steady-state is achieved relatively quickly, which is why this is the most popular type of analysis. Our approach has yielded *both* responses, together. In order to consider the steady-state only, all we must do is ignore the exponentially decaying terms, which are the initial conditions' contributions to the transient response.

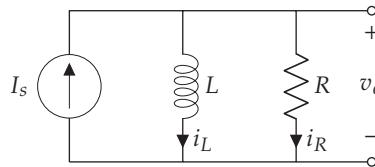
However, there are easier methods of obtaining the steady-state response if the transient response isn't of interest. The next chapter (chapter 3) considers these.

## 2.5 Problems



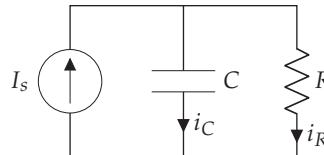
**Problem 2.1** Use the diagram below to answer the following questions and imperatives. Let  $I_s = A_0$ , where  $A_0 \in \mathbb{R}$  is a known constant. Perform a full circuit analysis, including the transient response. The initial inductor current is  $i_L(0) = 0$ .

- Write the elemental, KCL, and KVL equations.
- Write the differential equation for  $i_L(t)$  arranged in the standard form and identify the time constant  $\tau$ .
- Solve the differential equation for  $i_L(t)$  and use the solution to find the output voltage  $v_o(t)$ .

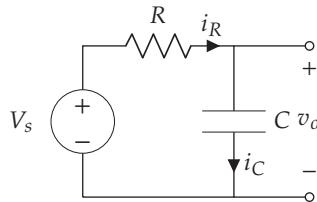


**Problem 2.2** Use the diagram below to answer the following questions and imperatives. Let  $I_s = A_0$ , where  $A_0 \in \mathbb{R}$  is a known constant. Perform a full circuit analysis, including the transient response. The initial capacitor voltage is  $v_C(0) = v_{C0}$ , a known constant.

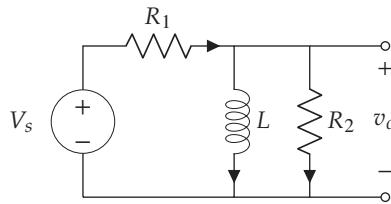
- Write the elemental, KCL, and KVL equations.
- Write the differential equation for  $v_C(t)$  arranged in the standard form.
- Solve the differential equation for  $v_C(t)$ .



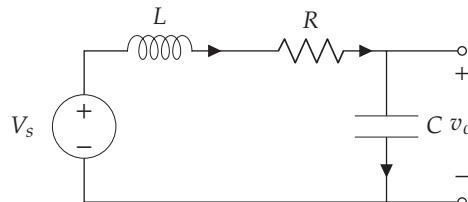
**Problem 2.3** For the RC circuit diagram below, perform a complete circuit analysis to solve for  $v_o(t)$  if  $V_S(t) = A \sin \omega t$ , where  $A \in \mathbb{R}$  is a given amplitude and  $\omega \in \mathbb{R}$  is a given angular frequency. Let  $v_C(t)|_{t=0} = v_{C0}$ , where  $v_{C0} \in \mathbb{R}$  is a given initial capacitor voltage. Hint: you will need to solve a differential equation for  $v_C(t)$ .



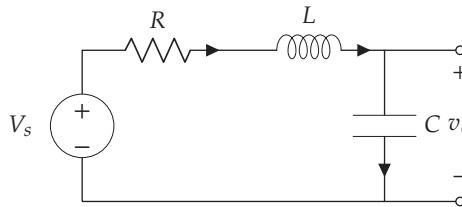
**Problem 2.4** **FRUITARIANISM** For the circuit diagram below, perform a complete circuit analysis to solve for  $v_o(t)$  if  $V_s(t) = A \sin \omega t$ , where  $A \in \mathbb{R}$  is a given amplitude and  $\omega \in \mathbb{R}$  is a given angular frequency. Let  $i_L(t)|_{t=0} = 0$  be the initial inductor current. Hint: you will need to solve a differential equation for  $i_L(t)$ .



**Problem 2.5** **GASTROLOBIUM** For the circuit diagram below, perform a complete circuit analysis to solve for  $v_o(t)$  if  $V_s(t) = 0$ . Let  $v_C(t)|_{t=0} = 5$  V and  $dv_C/dt|_{t=0} = 0$  V/s be the initial conditions. Assume the characteristic equation has distinct roots. Recommendation: due to the initial conditions being given in it, solve the differential equation in  $v_C$ .

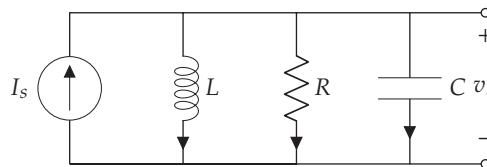


**Problem 2.6** **THYROPRIVIC** For the circuit diagram below, perform a complete circuit analysis to solve for  $v_o(t)$  if  $V_s(t) = 3 \sin(10t)$ . Let  $v_C(t)|_{t=0} = 0$  V and  $dv_C/dt|_{t=0} = 0$  V/s be the initial conditions. Assume the characteristic equation has distinct, complex roots. Recommendation: due to the initial conditions being given in it, solve the differential equation in  $v_C$ . Also, consider which, if any, of your results from problem 2.5 apply and re-use them, if so.



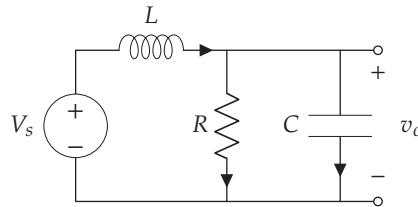
**Problem 2.7** **PHOTOCROMASCOPE** Use the circuit diagram below to answer the following questions and imperatives. Let  $I_s = A_0$ , where  $A_0 > 0$  is a known constant. Perform a full circuit analysis, including the transient response. The initial inductor current is  $i_L(0) = 0$  and the initial capacitor voltage is  $v_C(0) = 0$ . Assume the damping ratio  $\zeta \in (0, 1)$ ; i.e. the system is underdamped and the roots of the characteristic equation are complex.

- Write the elemental, KCL, and KVL equations.
- Write the second-order differential equation for  $i_L(t)$  arranged in the standard form and identify the natural frequency  $\omega_n$  and damping ratio  $\zeta$ .
- Convert the initial condition in  $v_C$  to a *second* initial condition in  $i_L$ .
- Solve the differential equation for  $i_L(t)$  and use the solution to find the output voltage  $v_o(t)$ . It is acceptable to use a known solution and to express your solution in terms of  $\omega_n$  and  $\zeta$ .



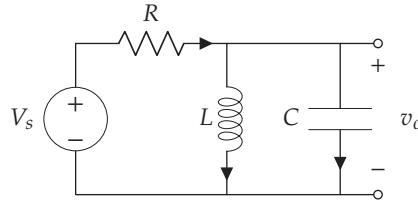
**Problem 2.8** **STADIUM** Use the circuit diagram below to answer the following questions and imperatives. Let  $V_s = A_0$ , where  $A_0 > 0$  is a known constant. The initial capacitor voltage is  $v_C(0) = 0$  and the initial inductor current is  $i_L(0) = 0$ .

- Write the elemental, KCL, and KVL equations.
- Derive the second-order differential equation in  $v_C(t)$ .
- Write the differential equation in standard form and identify the natural frequency  $\omega_n$  and damping ratio  $\zeta$ .
- Convert the initial condition in  $i_L$  to a *second* initial condition in  $v_C$ .
- Do not solve the differential equation*, but outline the steps you would take to solve it.



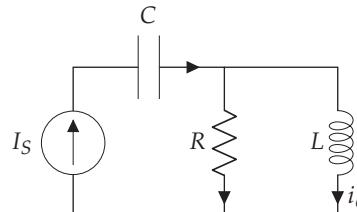
**Problem 2.9** Use the circuit diagram below to answer the following questions and imperatives. Let  $V_s = A_0$ , where  $A_0 > 0$  is a known constant. The initial capacitor voltage is  $v_C(0) = 5$  V and the initial inductor current is  $i_L(0) = 0$ .

- Write the elemental, KCL, and KVL equations.
- Derive the second-order differential equation in  $v_C(t)$ .
- Write the differential equation in standard form and identify the natural frequency  $\omega_n$  and damping ratio  $\zeta$ .
- Convert the initial condition in  $i_L$  to a *second* initial condition in  $v_C$  (i.e.,  $v'_C(0)$ ).
- Do not solve the differential equation*, but outline the steps you would take to solve it.



**Problem 2.10** Use the circuit diagram below to answer the following questions and imperatives. Let  $I_S = A_0$ , where  $A_0 > 0$  is a known constant. The initial capacitor voltage is  $v_C(0) = 0$  V and the initial inductor current is  $i_L(0) = 5$  A.

- Write the elemental, KCL, and KVL equations.
- Derive the first-order differential equation in  $i_L(t)$ .
- Write the differential equation in standard form with a time constant  $\tau$ .
- Solve the differential equation (find the specific solution) for  $i_o(t) = i_L(t)$ .





# 3

## Steady-State Circuit Analysis for Sinusoidal Inputs



Steady-state circuit analysis for sinusoidal inputs does not require the, at times, lengthy process of solving differential equations. Impedance methods, presented in this chapter, are shortcuts to steady-state analysis. It is important to note that impedance methods do not give information about the transient response.

### 3.1 Complex or Phasor Representations of Voltage and Current



It is common to represent voltage and current in circuits as complex exponentials, especially when they are sinusoidal. **Euler's formula** is our bridge back-and forth from trigonometric form ( $\cos \theta$  and  $\sin \theta$ ) and exponential form ( $e^{j\theta}$ ):

$$e^{j\theta} = \cos \theta + j \sin \theta.$$

Here are a few useful identities implied by Euler's formula.

$$e^{-j\theta} = \cos \theta - j \sin \theta$$

$$\cos \theta = \Re(e^{j\theta})$$

$$= \frac{1}{2} (e^{j\theta} + e^{-j\theta})$$

$$\sin \theta = \Im(e^{j\theta})$$

$$= \frac{1}{j2} (e^{j\theta} - e^{-j\theta}).$$

These equations can be considered to be describing a vector in the **complex plane**, which is illustrated in figure 3.1. Note that a  $e^{j\theta}$  has both a *magnitude* and a *phase*.

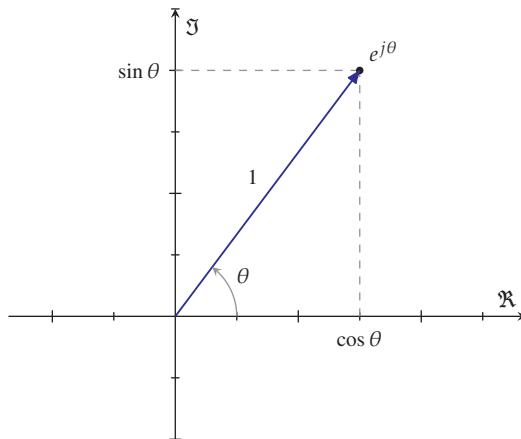


Figure 3.1. Euler's formula interpreted with a vector in the complex plane.

Consider a sinusoidal voltage signal  $v(t) = v_0 \cos(\omega t + \phi)$  with amplitude  $v_0$ , angular frequency  $\omega$ , and phase  $\phi$ . We encountered in section 2.4 the fact that, for a linear system with a sinusoidal input in steady-state, the output is a sinusoid at the same frequency as the input. The only aspects of the sinusoid that the system changed from input to output were its *magnitude* (amplitude) and *phase*. Therefore, these are the two quantities of interest in a steady-state circuit analysis. Our notation simply ignores the frequency  $\omega$  and represents  $v(t)$  as

$$v(t) = v_0 e^{j\phi}.$$

We call this the *complex* or **phasor form** of  $v(t)$ .

This is meant to be shorthand notation and, if interpreted literally, can cause confusion. In fact, mathematically,

$$\begin{aligned} v(t) &= v_0 \cos(\omega t + \phi) \\ &= v_0 \Re(e^{j(\omega t + \phi)}). \end{aligned}$$

Technically, we can use this more complicated form in our analysis but we won't because, conveniently, if we just treat the signal *as if it was equal to*  $v_0 e^{j\phi}$ , and at the end apply our "implied"  $e^{j\omega t}$  term and  $\Re()$  to the result, everything works out. This process is summarized in figure 3.2.

$$\begin{array}{ccc}
 v(t) = v_0 \cos(\omega t + \phi) & & v(t) = v'_0 \cos(\omega t + \phi') \\
 \text{phaze it!} \downarrow & & \uparrow \Re(e^{j\omega t}) \\
 v(t) = v_0 e^{j\phi} & \xrightarrow{\text{circuit operates}} & v'(t) = v'_0 e^{j\phi'}
 \end{array}$$

Figure 3.2. Summary of the use of phasors in circuit analysis.

The same process can be used to convert a sinusoidal current to and from phasor form. An alternative notation for a phasor  $v_0 e^{j\phi}$  is

$$v_0 \angle \phi.$$

### 3.1.1 Traversing Representations

Figure 3.3 shows transformations one might use to change signal representations. Often we begin with a trigonometric form and convert to phasor/polar form for analysis, which might require switching back and forth between phasor/polar and rectangular, depending on the operation:

- for *multiplication or division*, phasor/polar form is best and
- for *addition or subtraction* rectangular form is best.

Finally, it is often desirable to convert the result to trigonometric form, i.e. “dephaze” it.

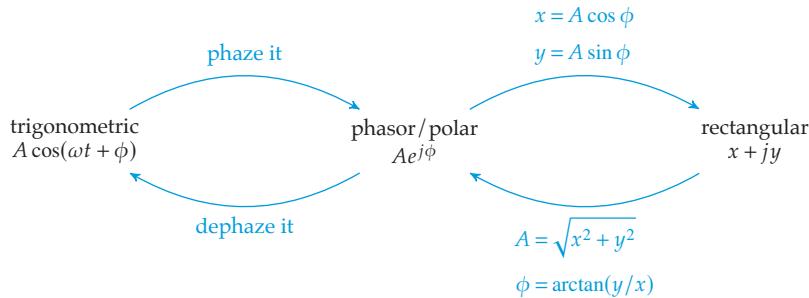


Figure 3.3. Transformations among trigonometric, phasor or polar, and rectangular forms of representation.

## 3.2 Impedance

With complex representations for voltage and current, we can introduce the concept of **impedance**.



### Definition 3.1

Impedance  $Z$  is the complex ratio of voltage  $v$  to current  $i$  of a circuit element:

$$Z = \frac{v}{i}.$$

The real part  $\Re(Z)$  is called the **resistance** and the imaginary part  $\Im(Z)$  is called the **reactance**. As with complex voltage and current, we can represent the impedance as a *phasor*.

Note that definition 3.1 is a generalization of Ohm's law. In fact, we call the following expression **generalized Ohm's law**:

$$v = iZ.$$

### 3.2.1 Impedance of Circuit Elements

The impedance of each of the three passive circuit elements we've considered thus far are listed, below. Wherever it appears,  $\omega$  is the angular frequency of the element's voltage and current.

**Resistor** For a resistor with resistance  $R$ , the impedance is all real:

$$Z_R = Re^{j0} = R.$$

**Capacitor** For a capacitor with capacitance  $C$ , the impedance is all imaginary:

$$Z_C = \frac{1}{\omega C} e^{-j\pi/2} = \frac{1}{j\omega C}.$$

**Inductor** For an inductor with inductance  $L$ , the impedance is all imaginary:

$$Z_L = \omega L e^{j\pi/2} = j\omega L.$$

These are represented in the complex plane in figure 3.4.

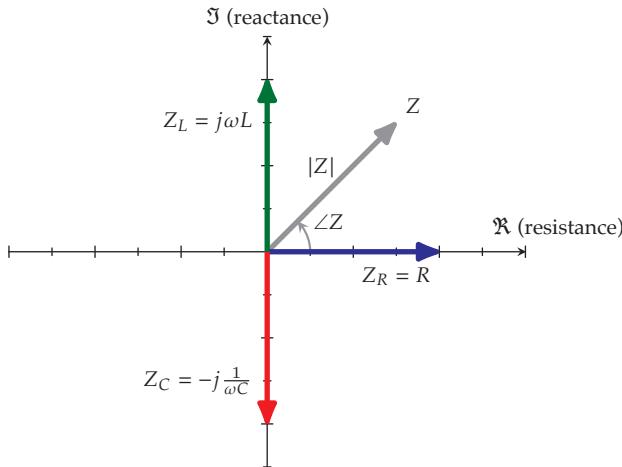


Figure 3.4. The impedances of a resistor  $Z_R$ , a capacitor  $Z_C$ , and an inductor  $Z_L$  in the complex plane.

### 3.2.2 Combining the Impedance of Multiple Elements

As with resistance, the impedance of multiple elements may be combined to find an **effective impedance**.

$K$  elements with impedances  $Z_j$  connected in *series* have equivalent impedance  $Z_e$  given by the expression

$$Z_e = \sum_{j=1}^K Z_j.$$

$K$  elements with impedances  $Z_j$  connected in *parallel* have equivalent impedance  $Z_e$  given by the expression

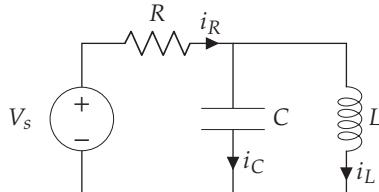
$$Z_e = \frac{1}{\sum_{j=1}^K 1/Z_j}.$$

In the special case of two elements with impedances  $Z_1$  and  $Z_2$ ,

$$\begin{aligned} Z_e &= \frac{1}{1/Z_1 + 1/Z_2} \\ &= \frac{Z_1 Z_2}{Z_1 + Z_2}. \end{aligned}$$

### Example 3.1

Given the circuit shown with voltage source  $V_s(t) = A e^{j\phi}$ , what is the total impedance at the source?



The total impedance is the combination of the series resistor impedance with the parallel capacitor and inductor impedances:

$$\begin{aligned}
 Z_e &= Z_R + \frac{Z_L Z_C}{Z_L + Z_C} \\
 &= R + \left( \omega L e^{j\pi/2} \frac{1}{\omega C} e^{-j\pi/2} \right) \left/ \left( j\omega L - j \frac{1}{\omega C} \right) \right. \\
 &= R + \frac{L/C}{j(LC\omega^2 - 1)/(\omega C)} \\
 &= R + \frac{L\omega}{j(LC\omega^2 - 1)} \\
 &= R - j \frac{L\omega}{LC\omega^2 - 1} \\
 &= A' e^{j\phi'}
 \end{aligned}$$

where we can compute the magnitude  $A'$  and phase  $\phi'$  to be

$$\begin{aligned}
 A' &= \sqrt{R^2 + \left( -\frac{L\omega}{LC\omega^2 - 1} \right)^2} \\
 \phi' &= \arctan \left( -\frac{L\omega}{LC\omega^2 - 1} \right) / R.
 \end{aligned}$$

Note that we used the fact that it's easier to multiply and divide phasors and add and subtract rectangular representations.



### 3.3 Methodology for Impedance-Based Circuit Analysis

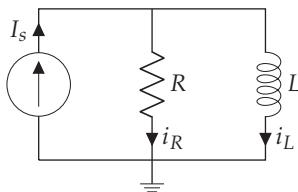
It turns out we can follow essentially the same algorithm presented in section 2.2 for analyzing circuits in steady-state with impedance. There are enough variations that we re-present it here.

Let  $n$  be the number of passive circuit elements in a circuit, which gives  $2n$  ( $v$  and  $i$  for each element) unknowns. The method is this.

1. Draw a *circuit diagram*.
2. Label the circuit diagram with the *sign convention* by labeling each element with the “assumed” direction of current flow.
3. Write *generalized Ohm’s law* for each circuit element and define the impedance of each element.
4. For every node not connected to a voltage source, write Kirchhoff’s current law (KCL).
5. For each loop not containing a current source, write Kirchhoff’s voltage law (KVL).
6. You probably have a linear system of  $2n$  *algebraic* equations (and  $2n$  unknowns) to be solved simultaneously. If only certain variables are of interest, these can be found by eliminating other variables such that the remaining system is smaller. The following steps can facilitate this process.
  1. Eliminate  $n$  (half) of the unknowns by substitution into the elemental equations (generalized Ohm’s law equations).
  2. Try substitution to eliminate to get down to only those variables of interest and inputs.
  3. Solve the remaining system of linear algebraic equations for the unknowns of interest.

#### Example 3.2

Given the RL circuit shown with current input  $I_s(t) = A \sin \omega t$ , what are  $i_L(t)$  and  $v_L(t)$  in steady-state?. Note that this is very similar to example 2.2, but we will use impedance methods.



1. The circuit diagram is given.
2. The signs are given.

3. The  $n$  elemental equations are as follows

$$L \quad v_L = i_L Z_L \quad (3.1)$$

$$R \quad v_R = i_R Z_R \quad (3.2)$$

where  $Z_L = j\omega L$  and  $Z_R = R$ .

4. There are actually two nodes not connected to the voltage source, but they give the same KCL equation

$$i_R = I_s - i_L.$$

5. There is one loop that doesn't have a current source in it, for which the KVL equation is

$$v_L = v_R.$$

6. Solve.

1. Eliminate  $v_L$  and  $i_R$  using KCL and KVL to yield the following.

$$L \quad i_L = v_R / Z_L \quad (3.3)$$

$$R \quad v_R = (I_s - i_L) Z_R \quad (3.4)$$

2. Substituting the  $R$  equation into the  $L$  equation, we eliminate  $v_R$  to obtain

$$i_L = \frac{Z_R}{Z_L} (I_s - i_L).$$

3. Solving,

$$i_L = \frac{Z_R}{Z_R + Z_L} I_s.$$

All that remains is to substitute, noting that we're using the cosine form of the phasor,

$$\begin{aligned}
 i_L &= \frac{R}{R + j\omega L} Ae^{-j\pi/2} \\
 &= \frac{Re^{j0}}{M_1 e^{j\phi_1}} Ae^{-j\pi/2} \\
 &= \frac{R}{M_1} Ae^{-j\phi_1} e^{-j\pi/2} \\
 &= \frac{R}{M_1} Ae^{-j(\phi_1 + \pi/2)} \\
 &= \frac{R}{M_1} A \cos(\omega t - \phi_1 - \pi/2) \\
 &= \frac{R}{M_1} A \sin(\omega t - \phi_1)
 \end{aligned}$$

where

$$\begin{aligned}
 M_1 &= \sqrt{R^2 + (L\omega)^2} \text{ and} \\
 \phi_1 &= \arctan(L\omega/R).
 \end{aligned}$$

Note that we could have used the unconventional sine form of the phasor. Also note that (i) this was somewhat easier than how we did it in example 2.2 and (ii) the result simply scales the amplitude and shifts the phase, as expected.

- Finally, we can find  $v_L(t)$  from the inductor elemental equation (generalized Ohm's law):

$$\begin{aligned}
 v_L(t) &= i_L Z_L \\
 &= \omega L e^{j\pi/2} \frac{R}{M_1} A e^{-j(\phi_1 + \pi/2)} \\
 &= \frac{RL\omega}{M_1} A e^{-j\phi_1} \\
 &= \frac{RL\omega}{M_1} A \cos(\omega t - \phi_1).
 \end{aligned}$$



### 3.4 Voltage and Current Dividers

In (lec:voltage\_dividers), we developed the useful voltage divider formula for quickly analyzing how voltage divides among series resistors. This can be considered a special case of a more general voltage divider equation for any elements described by an impedance. After developing the voltage divider, we also introduce the current divider, which divides an input current among parallel elements.

#### 3.4.1 Voltage Dividers

First, we develop the solution for the two-element voltage divider shown in figure 3.5. We choose the voltage across  $Z_2$  as the output. The analysis can follow our usual methodology of six steps, solving for  $v_2$ .

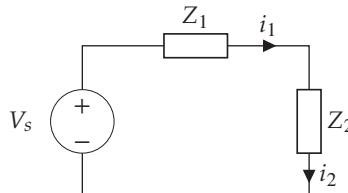


Figure 3.5. The two-element voltage divider.

1. The circuit diagram is given in figure 3.5.
2. The assumed directions of positive current flow are given in figure 3.5.
3. The elemental equations are just generalized Ohm's law equations.

$$Z_1 \mid v_1 = i_1 Z_1$$

$$Z_2 \mid v_2 = i_2 Z_2$$

4. The KCL equation is  $i_2 = i_1$ .
5. The KVL equation is  $v_1 = V_s - v_2$ .
6. Solve.

1. Eliminating  $i_2$  and  $v_1$  from KCL and KVL, our elemental equations become the following.

$$Z_1 \mid i_1 = v_1/Z_1 = (V_s - v_2)/Z_1$$

$$Z_2 \mid v_2 = i_1 Z_2$$

2. Eliminating  $i_1$ ,

$$v_2 = \frac{Z_2}{Z_1} (V_s - v_2).$$

3. Solving for  $v_2$ ,

$$v_2 = \frac{Z_2}{Z_1 + Z_2} V_s.$$

A similar analysis can be conducted for  $n$  impedance elements.

### Definition 3.2: Impedance Voltage Divider

For the output voltage across impedance  $Z_k$  in series with  $n$  impedance elements with input  $v_{\text{in}}$  is

$$v_k = \frac{Z_k}{Z_1 + Z_2 + \dots + Z_k + \dots + Z_n} v_{\text{in}}.$$

### 3.4.2 Current Dividers

By a similar process, we can analyze a circuit that divides current into  $n$  parallel impedance elements.

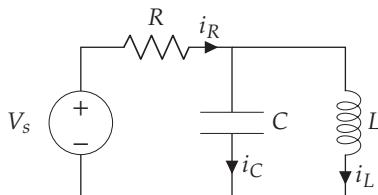
### Definition 3.3: Impedance Current Divider

For the output current through impedance  $Z_k$  in parallel with  $n$  impedance elements with input current  $i_{\text{in}}$  is

$$i_k = \frac{1/Z_k}{1/Z_1 + 1/Z_2 + \dots + 1/Z_k + \dots + 1/Z_n} i_{\text{in}}.$$

### Example 3.3

Given the circuit shown with voltage source  $V_s(t) = A e^{j\phi}$  and output  $v_L$ , what is the ratio of output over input amplitude? What is the phase shift from input to output?



We'll use a voltage divider:

$$\begin{aligned}
 v_L &= \frac{\frac{Z_L Z_C}{Z_L + Z_C}}{Z_R + \frac{Z_L Z_C}{Z_L + Z_C}} V_s \quad \Rightarrow \\
 \frac{v_L}{V_s} &= \frac{Z_L Z_C}{Z_R Z_L + Z_R Z_C + Z_L Z_C} \\
 &= \frac{L/C}{jRL\omega - jR/(C\omega) + L/C} \\
 &= \frac{L\omega}{jRLC\omega^2 - jR + L\omega} \\
 &= \frac{L\omega}{L\omega + jR(LC\omega^2 - 1)} \\
 &= \underbrace{\frac{L\omega}{\sqrt{(L\omega)^2 + R^2(LC\omega^2 - 1)^2}}}_{\text{magnitude ratio}} \exp \underbrace{\left( -\arctan \frac{R(LC\omega^2 - 1)}{L\omega} \right)}_{\text{phase difference}}.
 \end{aligned}$$

---

### 3.5 Problems



**Problem 3.1** For the RC circuit diagram below, perform a circuit analysis to solve for the steady state voltage  $v_o(t)$  if  $V_S(t) = A \sin \omega t$ , where  $A \in \mathbb{R}$  is a given amplitude and  $\omega \in \mathbb{R}$  is a given angular frequency. Use a *sine* phasor in the problem. Write your answer as a single sine phasor in *polar* form. Evaluate your answer for the following two sets of parameters.

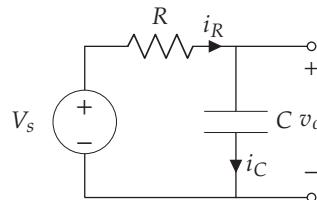
$$A = 2, 5 \text{ V}$$

$$\omega = 10 \times 10^3, 20 \times 10^3 \text{ rad/s}$$

$$R = 100, 1000 \Omega$$

$$C = 100, 10 \text{ nF.}$$

The first set should yield  $v_o = 1.99e^{-j0.0997}$ .



**Problem 3.2** For the circuit diagram below, perform a complete circuit analysis to solve for the steady state voltage  $v_o(t)$  if  $V_S(t) = A \sin \omega t$ , where  $A \in \mathbb{R}$  is a given amplitude and  $\omega \in \mathbb{R}$  is a given angular frequency. Use a *sine* phasor in the problem. Write your answer as a single sine phasor in *polar* form. Evaluate your answer for the following two sets of parameters.

$$A = 3, 8 \text{ V}$$

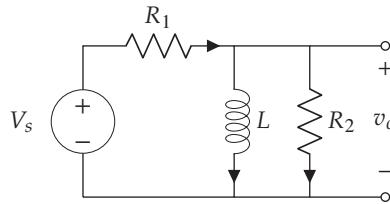
$$\omega = 30 \times 10^3, 60 \times 10^3 \text{ rad/s}$$

$$R_1 = 100, 1000 \Omega$$

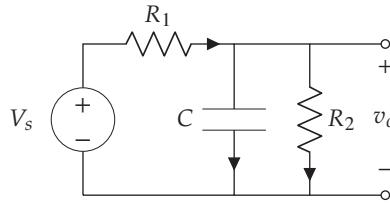
$$R_2 = 1000, 100 \Omega$$

$$L = 10, 100 \text{ mH.}$$

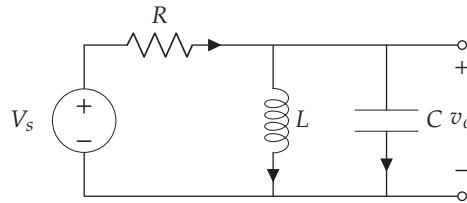
The first set should yield  $v_o = 2.61e^{j0.294}$ .



**Problem 3.3** For the circuit diagram below, solve for the steady state voltage  $v_o(t)$  if  $V_s(t) = A e^{j\phi}$ , where  $A \in \mathbb{R}$  is a given input amplitude and  $\phi \in \mathbb{R}$  is a given input phase. Write your answer as a single phasor in *polar* form (you may use intermediate variables in this final form as long as they're clearly stated).



**Problem 3.4** For the circuit diagram below, solve for the steady state output voltage  $v_o(t)$  if  $V_s(t) = A \cos(\omega t)$ . Do write  $V_s$  and the impedance of each element in phasor/polar form. Do not substitute  $V_s$  or the impedance of each element into your expression for  $v_o(t)$ . Recommendation: use a divider rule.



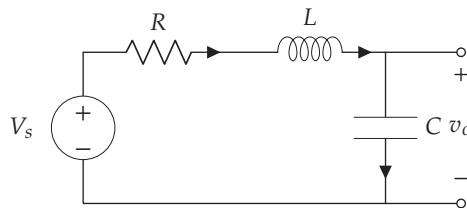
**Problem 3.5** For the circuit diagram below, solve for the steady state output voltage  $v_o(t)$  if  $V_s(t) = 3 \sin(10t)$ . Use a *sine* phasor in the problem. Write your answer as a single sine phasor in *polar* form. Evaluate your answer for the following two sets of parameters.

$$R = 10, 10^6 \Omega$$

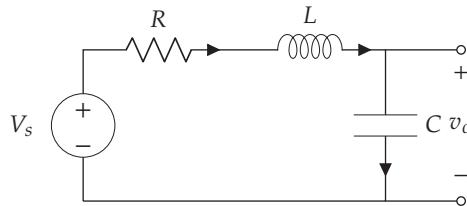
$$L = 500, 50 \text{ mH}$$

$$C = 100, 10 \mu\text{F}$$

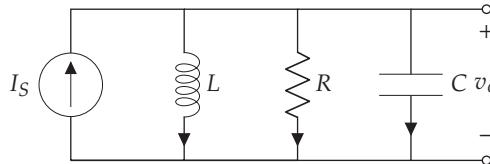
The first set should yield  $v_o = 3.01e^{-j0.0100}$ .



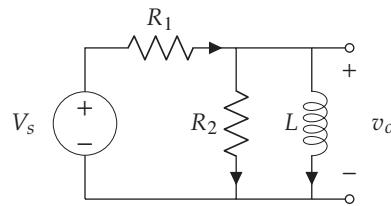
**Problem 3.6** For the circuit diagram below, solve for  $v_o(t)$  if  $V_s(t) = A \sin \omega t$ , where  $A = 2$  V is the given amplitude and  $\omega \in \mathbb{R}$  is a given angular frequency. Let  $R = 50 \Omega$ ,  $L = 50 \text{ mH}$ , and  $C = 200 \text{ nF}$ . Find the steady-state ratio of the output amplitude to the input amplitude  $A$  for  $\omega = \{5000, 10000, 50000\} \text{ rad/s}$ . Plot the steady-state ratio as a function of  $\omega$  in MATLAB, Python, or Mathematica. This circuit is called a **low-pass filter**—explain why this makes sense. Note that using impedance methods for steady state analysis makes this problem much easier than the transient analysis of this circuit in Exercise 2.6.



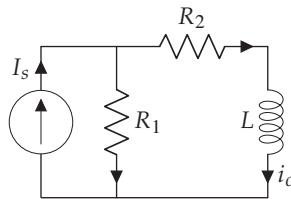
**Problem 3.7** For the circuit diagram below, perform a circuit analysis to solve for the steady state voltage  $v_o(t)$  if  $I_s = Ae^{j0}$ , where  $A > 0$  is a given amplitude. Identify all impedance values in the circuit, but express your answer in terms of impedances (i.e. don't substitute for them in your final expression).



**Problem 3.8** For the circuit diagram below, perform a complete circuit analysis to solve for the steady state voltage  $v_o(t)$  if  $V_s(t) = Ae^{j\phi}$ , where  $A \in \mathbb{R}$  is a given input amplitude and  $\phi \in \mathbb{R}$  is a given input phase. Write your answer as a single phasor in *polar form*. *Hint:* consider using a divider rule, but be wary of the parallel impedances for  $R_2$  and  $L$ !



**Problem 3.9**  For the circuit diagram below, perform a complete circuit analysis to solve for the steady state current  $i_o(t)$  if  $I_s(t) = Ae^{j\phi}$ , where  $A \in \mathbb{R}$  is a given input amplitude and  $\phi \in \mathbb{R}$  is a given input phase. Write your answer as a single phasor in *polar* form. *Hint:* consider using the current divider rule, but be wary of the series impedances for  $R_2$  and  $L$ ! Clearly define any new constants you introduce.



# 4

## Nonlinear and Multiport Circuit Elements

5P



Thus far, we have considered only *one-port, linear* circuit elements. One-port elements have two terminals. Linear elements have voltage-current relationships that can be described by linear algebraic or differential equations.

**Multi-port elements** are those that have more than one port. In this chapter, we will consider several multi-port elements: transformers (two-port), transistors (two-port), and opamps (four-port).

**Nonlinear elements** have voltage-current relationships that cannot be described by a linear algebraic or differential equations. The convenient impedance methods of chapter 3 apply only to linear circuits, so we must return to the differential equation-based analysis of chapter 2. In this chapter, we will consider several nonlinear circuits containing three different classes of nonlinear elements: diodes, transistors, and opamps.

A great number of the most useful circuits today include multi-port and nonlinear elements. Tasks such as ac-dc conversion, switching, amplification, and isolation require these elements.

We explore only the fundamentals of each element considered and present basic analytic techniques, but further exploration in Horowitz and Hill (2015), Agarwal and Lang (2005), and Ulaby, Maharbiz, and Furse (2018) is encouraged.



## 4.1 Transformers

Electrical transformers are *two-port linear* elements that consist of two tightly coupled coils of wire. Due to the coils' magnetic field interaction, time-varying current through one side induces a current in the other (and vice-versa).

Let the terminals on the **primary (source) side** have label “1” and those on the **secondary (load) side** have label “2,” as shown in figure 4.1. These devices are very efficient, so we often assume no power loss. With this assumption, the power into the transformer must sum to zero, giving us one voltage-current relationship:

$$\mathcal{P}_1 + \mathcal{P}_2 = 0$$

$$v_1 i_1 = -v_2 i_2.$$

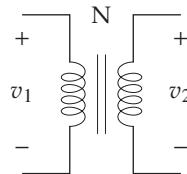


Figure 4.1. Circuit symbol for a transformer with a core. Those with “air cores” are denoted with a lack of vertical lines.

Note that with two ports, we need two elemental equations to fully describe the voltage-current relationships. Another equation can be found from the magnetic field interaction. Let  $N_1$  and  $N_2$  be the number of turns per coil on each side and  $N \equiv N_2/N_1$ . Then

$$\frac{v_1}{v_2} = \frac{1}{N}.$$

These two equations can be combined to form the following elemental equations.

### Definition 4.1

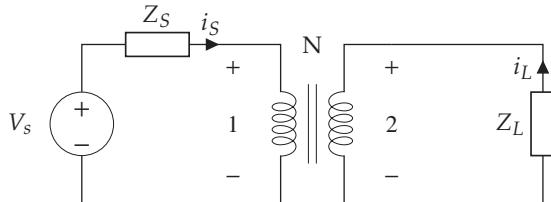
$$v_2 = N v_1 \quad i_2 = -\frac{1}{N} i_1$$

So we can **step-down** voltage if  $N < 1$ . This is better, in some cases, than the voltage divider because it does not dissipate much energy. However, transformers can be bulkier and somewhat nonlinear; moreover, they *only work for ac signals*. Note that when we step-down voltage, we step-up current due to our power conservation assumption.

If  $N > 1$  we can **step-up** voltage. Voltage dividers cannot do this! It is not amplification, however, because power is conserved—we simultaneously step-down current. So with a transformer, we can *freely trade ac voltage and current*.

### Example 4.1

Given the circuit shown, what is the effective impedance of  $Z_L$  on the source side?



Since the effective impedance on the primary side of the transformer is, by definition,

$$Z_1 = v_1 / i_1,$$

we can use the transformer elemental equations to write that in terms of the voltage and current on the secondary side:

$$\begin{aligned} Z_1 &= \frac{v_2/N}{-i_2 N} \\ &= -\frac{1}{N^2} \cdot \frac{v_2}{i_2}. \end{aligned}$$

Let's write that in terms of the voltage and current of the load:

$$Z_1 = \frac{1}{N^2} \cdot \frac{v_L}{i_L}.$$

From the definition of the impedance of an element,  $v_L / i_L = Z_L$ , so

$$Z_1 = \frac{1}{N^2} Z_L.$$

This is a useful result. We say that the transformer *reflects* the impedance  $Z_L$  through the factor  $1/N^2$ .



## 4.2 Diodes

Diodes are *single-port nonlinear* elements that, approximately, conduct current in only one direction. We will consider the ubiquitous **semiconductor diode**, varieties of which include the **light-emitting diode (LED)**, **photodiode** (for light sensing), **Schottky diode** (for fast switching), and **Zener diode** (for voltage regulation). See figure 4.2 for corresponding circuit symbols.

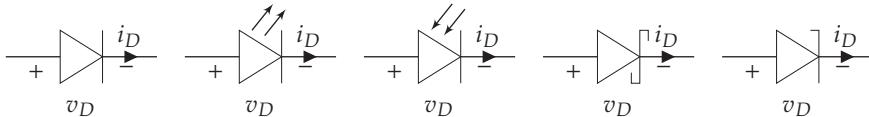


Figure 4.2. Diode symbols. From left to right, the generic symbol, LED, photodiode, Schottky, Zener.

In most cases, we use the diode to conduct current in one direction and block reverse current.<sup>1</sup> When conducting current in its forward direction, it is said to have **forward-bias**; when blocking current flow in its reverse direction, it is said to have **reverse-bias**. If the reverse **breakdown voltage** is reached, current will flow in the reverse direction. It is important to check that a circuit design does not subject a diode to its breakdown voltage, except in special cases (e.g. when using a Zener diode).

We begin with a nonlinear model of the voltage-current  $v_D - i_D$  relationship. Let

- $I_s$  be the *saturation current* (typically  $\sim 10^{-12}$  A) and
- $V_{TH} = k_b T / e$  be the *thermal voltage* (at room temperature  $\sim 25$  mV) with<sup>2</sup>
  - $k_b$  the Boltzmann constant,
  - $e$  the fundamental charge, and
  - $T$  the diode temperature.

### Definition 4.2

Let the nonlinear diode model be

$$i_D = I_s (\exp(v_D/V_{TH}) - 1).$$

See figure 4.3 for a plot of this function. One can analyze circuits with diodes using the methods of chapter 2 and definition 4.2 as the diode's elemental equation.

1. The paradigmatic exception is the Zener diode, which is typically used in reverse bias in order to take advantage of its highly stable reverse bias voltage over a large range of reverse current. We will not consider this application here.

2. Unless otherwise specified, it is usually reasonable to assume room-temperature operation.

A nonlinear set of equations results, which typically require numerical solution techniques.

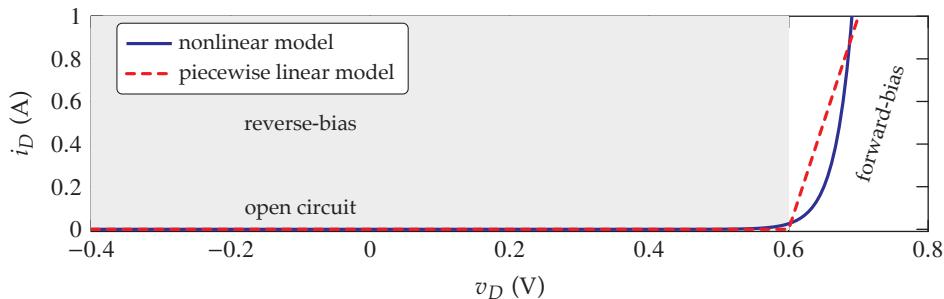


Figure 4.3. The voltage-current relationship in the nonlinear and piecewise linear models. In the figure,  $R_d = 0.1 \Omega$ .

#### 4.2.1 A Piecewise Linear Model

An **ideal diode** is one that is a perfect insulator (open circuit,  $i_D = 0$ ) for  $v_D < 0$  and a perfect conductor for  $v_D > 0$ . We use the symbol shown in figure 4.4 for the ideal diode. At times, the ideal diode is sufficient to model a diode; often, however, we prefer a more accurate model that is piecewise linear.

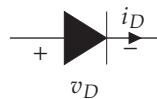


Figure 4.4. Circuit symbol for an ideal diode. Note that this is a nonstandard use of this symbol.

The **piecewise linear model** is shown in figure 4.5. It includes an ideal diode in series with a fixed voltage drop of 0.6 V and a resistor with resistance  $R_d$ . This approximates the nonlinear model with two linear approximations.

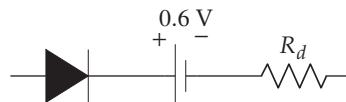


Figure 4.5. Piecewise linear diode.

### Definition 4.3

Let a piecewise linear diode model be

$$i_D = \begin{cases} 0 & \text{for } v_D \leq 0.6 \text{ V} \\ \frac{v_D - 0.6 \text{ V}}{R_d} & \text{for } v_D > 0.6 \text{ V} \end{cases}$$

See figure 4.3 for a plot of this function and a comparison to the nonlinear model.

The slope in forward-bias is  $1/R_d$ . This model's effectiveness is highly dependant on  $R_d$ , so an **operating point** must be chosen and  $R_d$  chosen to match most closely with the nonlinear model near that operating point.

### 4.2.2 Method of Assumed States

The **method of assumed states** is a method for using linear circuit analysis to analyze circuits with nonlinear components. The method is summarized in the following steps.

1. Begin at the initial time  $t = 0$ .
2. Replace each diode in the circuit diagram with the piecewise linear diode model.
3. Proceed with the circuit analysis of chapter 2, ignoring the elemental equations for the ideal diodes  $D_i$ . Your system of equations will have unknown ideal diode current  $i_{D_i}$  and voltage  $v_{D_i}$ . Simplify it to the extent possible.
4. Guess the current state of each ideal diode: ON or OFF. For each ideal diode  $D_i$  guessed to be ON,

$$\text{set } v_{D_i} = 0 \text{ and assume that } i_{D_i} > 0.$$

For each ideal diode assumed to be OFF,

$$\text{set } i_{D_i} = 0 \text{ and assume that } v_{D_i} < 0.$$

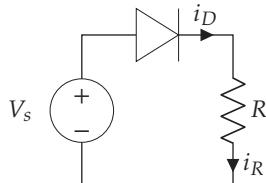
For  $n$  diodes in the circuit, there are  $2^n$  possibilities at each moment in time. Guess just one to start.

5. If *even one diode* violates its assumption from above, dismiss the results and return to step 4 and choose a different combination of assumed states (consider flipping the assumptions on those diodes that violated the old assumptions).
6. If *not even one diode* violates its assumptions, *this is the correct solution for this moment in time*.
7. This solution is valid for as long as its assumptions are valid. Once they fail, go back to step 4.

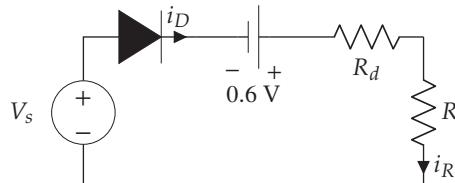
Since impedance methods are valid only for *linear* circuits, steady-state analyses should proceed with the same process outlined above. With a periodic input, a periodic (steady) solution may emerge.

### Example 4.2

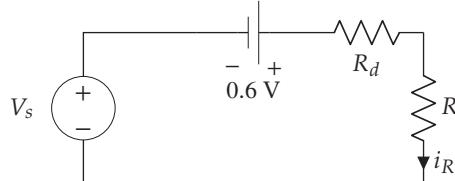
Given the circuit shown with voltage source  $V_s(t) = 3 \cos 2\pi t$ , what is the output  $v_R$ ? Explain why this might be called a "half-wave rectifier." Let  $R = 10 \Omega$ .



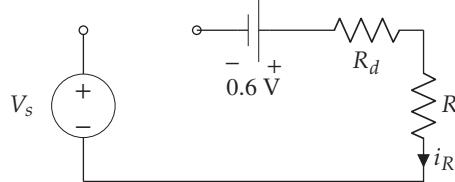
Let's use the piecewise linear model, which can be drawn as shown. We use the method of assumed states. First, subcircuits.



At right, we have the subcircuit with the *on* state of the ideal diode.



At right, we have the subcircuit with the *off* state of the ideal diode.



Now, to analyze each circuit. The on circuit, if we consider the dc voltage source to be an offset of the ac source, it is a voltage divider:

$$v_R = \frac{R}{R + R_d} (V_s - 0.6).$$

The value of  $R_d$  is tbd. Let's use a thought experiment to decide what range of current for which we might need  $R_d$  to be valid. We know that

$$\begin{aligned} i_D &= i_R \\ &= v_R/R. \end{aligned}$$

If  $R_d = 0$ , the maximum value of  $v_R$  is  $3 - 0.6 = 2.4$  V. Therefore,

$$\begin{aligned} i_{D\max} &= 2.4/10 \\ &= 0.24 \text{ A.} \end{aligned}$$

From figure 4.3, we can choose a reasonable value of  $R_d = 0.2 \Omega$ . Now that we have  $R_d$ , so we now have our on solution:

$$\begin{aligned} v_R &= \frac{10}{10 + 0.2}(3 \cos 2\pi t - 0.6) \\ &\approx 2.94 \cos 2\pi t - 0.588. \end{aligned}$$

The off solution for  $v_R$  is  $v_R = i_R R = (0)R = 0$  V. To determine which solution is valid, we need to determine when the on solution predicts  $i_D < 0$ . Since  $i_D = i_R = v_R/R$ ,

$$i_D = 0.294 \cos 2\pi t - 0.0588$$

which is positive for an interval each period. The endpoints of the interval correspond to

$$\begin{aligned} 0 &= 0.294 \cos 2\pi t - 0.0588 \\ t &= \pm \frac{1}{2\pi} \arccos 0.2 \pm T \\ &= \pm 0.218 \pm T \text{ sec} \end{aligned}$$

where  $T \in \mathbb{N}_0$  is the period. The function is positive between these points in each period. Therefore, for all positive intervals, the on solution is correct, and for all negative intervals, the off solution is correct.

Consider figure 4.6. The voltage output is always positive, but the effects of the non-ideality of the diode are apparent with the voltage offset and the (slight) scaling.

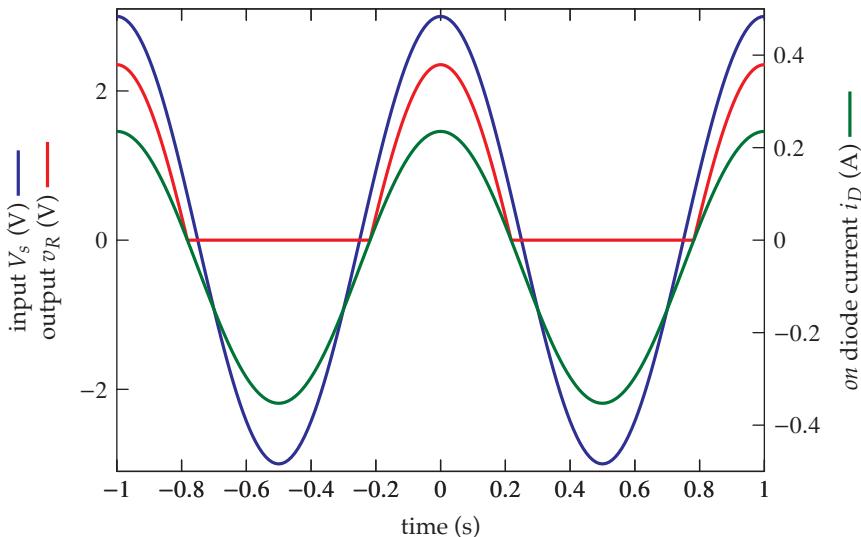


Figure 4.6

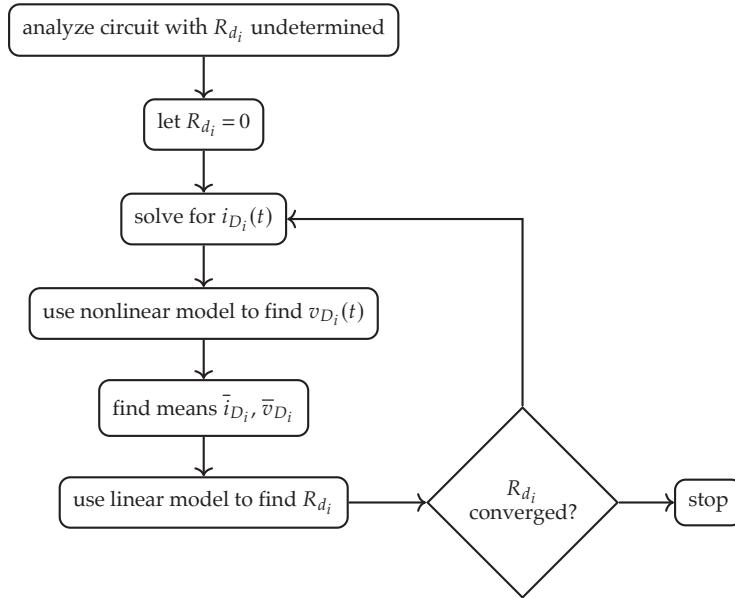
This circuit is called the half-wave rectifier because it gives approximately the upper-half of the wave at its output. This introduces a dc offset. With some filtering, this can be used as an ac-to-dc rectifier. Note that typically a *full-wave bridge rectifier* is used for this application.

### 4.2.3 An Algorithm for Determining $R_d$

The piecewise linear approximation of the exponential diode current will never be great, but we can at least try to choose  $R_d$  in a somewhat optimal way, recognizing that when highly accurate results are required, there's no substitute for the nonlinear model.

Consider the algorithm of figure 4.7. Initially set to zero the diode resistances  $R_{d_i}$  of each resistor. Solve for each diode current  $i_{D_i}(t)$ , then use this to find  $v_{D_i}(t)$  from the *nonlinear* model of definition 4.2:

$$v_{D_i}(t) = V_{TH} \ln(i_{D_i}(t)/I_s + 1).$$

Figure 4.7. An algorithm for determining  $R_{d_i}$ .

Now take the means of these signals (assuming steady state oscillation) over a period  $T$ , excluding the time  $T_0$  during which the diode voltage was in reverse-bias:<sup>3</sup>

$$\bar{i}_{D_i} = \frac{1}{T - T_0} \int_{t_0}^{t_0+T} i_{D_i}(\tau) d\tau \quad (4.1)$$

$$\bar{v}_{D_i} = \frac{1}{T - T_0} \int_{t_0}^{t_0+T} v_{D_i}(\tau) d\tau. \quad (4.2)$$

Now us the piecewise linear model of definition 4.3 to estimate  $R_{d_i}$ :

$$R_{d_i} = \frac{\bar{v}_{D_i} - 0.6V}{\bar{i}_{D_i}}.$$

We can use this estimate of  $R_{d_i}$  to re-analyze the circuit and repeat the same process of deriving a new estimate of  $R_{d_i}$ . This process should converge on an estimate of  $R_{d_i}$  that is in some sense optimal.

Note that if, during this iterative process, one finds  $\bar{v}_{D_i} < 0.6$  V, a negative  $R_d$  will result. At this point, a couple different reasonable approaches can be taken:

3. Note that if  $T_0$  is ignored, our estimate of  $R_d$  will include the effects of time during which no current is flowing and the diode is in reverse-bias, during which time  $R_d$  is not applicable.

1. just use  $R_{d_i} = 0$  or
2. use some reasonably central value of  $\bar{v}_{D_i} > 0.6$  V.

The second case is preferred if  $v_{D_i}(t)$  spends much time above 0.6 V. But usually, if it spends much time, the mean  $\bar{v}_{D_i}$  should be great enough to avoid this situation. Circuits that tend to express this behavior are those with high impedance and correspondingly low currents.

### 4.3 MOSFETs

A metal–oxide–semiconductor field-effect transistor (MOSFET) is a *two-port, nonlinear* circuit element that lies at the heart of digital electronics, with sometimes millions integrated into a single microprocessor. They are the dominant type of **transistor**, a class of elements that includes the **bipolar junction transistor (BJT)**.



MOSFETs are not just common in integrated circuits made of silicon, they are also available as discrete elements, which is the form most often encountered by the mechatronicist.

There are two primary types of MOSFET: the **n-channel** and the **p-channel**, determined by the type of semiconductor doping (negative or positive) used in the manufacturing process. These types are “opposites,” so we choose to focus on n-channel, here.

Figure 4.8 displays the circuit diagram symbol for the MOSFET. There are three<sup>4</sup> terminals: the **gate**  $G$ , **drain**  $D$ , and **source**  $S$ . The current flowing from one terminal to another is labeled with consecutive subscripts; for instance, the current flowing from drain to source is  $i_{DS}$ . Similarly, the voltage drop across two terminals is labeled with concurrent subscripts; for instance, the voltage drop from gate to source is  $v_{GS}$ .

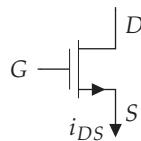


Figure 4.8. Circuit symbol for a n-channel MOSFET.

The input-output characteristics of the MOSFET are quite complex, but we may, in the first approximation, consider it to be like a *switch*. In this model, called the

4. Note that if we consider the gate-side to be the input with  $i_{GS} = 0$  and  $v_{GS}$  and the drain-source-side to be the output with  $i_{DS}$  and  $v_{DS}$ , the MOSFET can be seen to be two-port.

**S-model**, if the gate voltage  $v_{GS}$  is less than the **threshold voltage**  $V_T$  (typically around 0.7 V), the  $D$  and  $S$  terminals are disconnected (open) from each other (OFF mode). But when  $v_{GS} > V_T$ ,  $D$  and  $S$  are connected via a short and current  $i_{DS}$  can flow (ON mode).

The input-output characteristics of a MOSFET are actually much more complex than the S-model captures. The S-model can build intuition and suffice for digital logic circuit analysis. However, we are here mostly concerned with analog circuit models. Specifically, we mechatronicists use MOSFETs to drive power-hungry loads (e.g. motors) with high-power sources controlled by low-power microcontrollers. We now turn to a general model, after which we consider a method of analyzing MOSFET circuits.

### 4.3.1 The Switch Unified (SU) Model

The **switch unified (SU) model** is reasonably accurate at describing actual MOSFET input-output characteristics. However, it is quite *nonlinear*, and therefore can give us headaches during analysis. As usual, we are concerned with the element's voltage-current relationships.

#### Definition 4.4

Let  $K$  be a constant parameter of the MOSFET with units  $\text{A/V}^2$ .  $K$  can be found from parameters of a given MOSFET. The current into the gate is zero:  $i_G = 0$ . The current from drain to source is controlled by the two voltage variables  $v_{GS}$  and  $v_{DS}$ , as shown.

$$i_{DS} = \begin{cases} 0 & \text{for } v_{GS} < V_T \\ K((v_{GS} - V_T)v_{DS} - v_{DS}^2/2) & \text{for } v_{GS} \geq V_T \text{ and } v_{DS} < v_{GS} - V_T \\ \frac{K}{2}(v_{GS} - V_T)^2 & \text{for } v_{GS} \geq V_T \text{ and } v_{DS} \geq v_{GS} - V_T \end{cases}$$

So, as in the S-model, the gate voltage  $v_{GS}$  must exceed the threshold voltage  $V_T$  for current to flow. The interval below the threshold is called the **cutoff region** (OFF). Note, however, that current doesn't just flow freely, as it would with the short of the S-model. In fact, two distinct ON ( $v_{GS} > V_T$ ) intervals emerge. In both, the current  $i_{DS}$  depends on  $v_{GS}$ . In the **triode region**,  $v_{DS} < v_{GS} - V_T$ ,  $i_{DS}$  also depends on  $v_{DS}$ . However, in the **saturation region**,  $v_{DS} > v_{GS} - V_T$ ,  $i_{DS}$  is independent of  $v_{DS}$  and can be controlled by  $v_{GS}$ , alone.

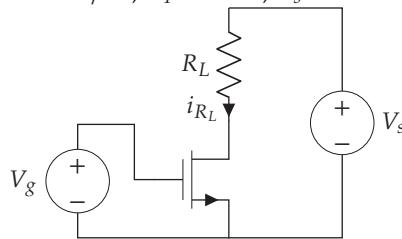
Note that in saturation, the MOSFET behaves like a current source controlled by  $v_{GS}$ . A source controlled by a variable in the circuit is called a **dependent source**. This behavior as a dependent current source (that can also be turned off) is the most valuable for us.

The **switch current source (SCS) model** is actually just a recognition of this behavior and an elimination of the triode region from consideration. This is a reasonable assumption if we can guarantee operation in cutoff or saturation only.

Given the piecewise MOSFET models, we can again use the **method of assumed states** for MOSFET circuit analysis. Note however that only the S-model is piecewise linear and that the SU- and SCS-models are piecewise nonlinear. We can handle some relatively simple nonlinear cases analytically, but require either linearization or numerical assistance for more complex circuit analyses.

### Example 4.3

Given the circuit shown, solve for the voltage across the load  $R_L$  for varying  $V_g$  given the following conditions: saturation of the MOSFET,  $R_L = 1 \text{ k}\Omega$ ,  $K = 0.5 \text{ mA/V}^2$ ,  $V_T = 0.7 \text{ V}$ ,  $V_s = 10 \text{ V}$ .



During saturation,

$$i_{DS} = \frac{K}{2}(v_{GS} - V_T)^2.$$

We know that  $v_{GS} = V_g$  and the other parameters are given. The voltage across the resistor is, from Ohm's law,

$$v_{RL} = i_{RL} R_L.$$

From KCL,

$$\begin{aligned} v_{RL} &= i_{DS} R_L \\ &= \frac{KR_L}{2}(V_g - V_T)^2. \end{aligned}$$

This is our load voltage, but we must determine the range of  $V_g$  for which saturation is satisfied. We must first find  $v_{DS}$ . From KVL,

$$\begin{aligned} v_{DS} &= V_s - v_{RL} \\ &= V_s - \frac{KR_L}{2}(V_g - V_T)^2. \end{aligned}$$

The condition for saturation (in addition to  $V_g > V_T$ ) is

$$v_{DS} \geq v_{GS} - V_T \Rightarrow$$

$$V_s - \frac{KR_L}{2}(V_g - V_T)^2 \geq V_g - V_T \Rightarrow$$

$$V_T - \frac{1 + \sqrt{2KR_LV_s}}{KR_L} \leq V_g \leq V_T + \frac{-1 + \sqrt{2KR_LV_s}}{KR_L}.$$

Since  $V_g > V_T$ , we can simplify the condition for saturation to

$$V_T < V_g \leq V_T + \frac{-1 + \sqrt{2KR_LV_s}}{KR_L}.$$

Computing the upper bound, we obtain 5.33 V. See figure 4.9 for the plot. Note that the similar input and load voltage might suggest we didn't need to use a MOSFET. However, the input voltage has virtually zero load, whereas the source voltage does. Essentially, the current draw (and therefore power) is different! We're controlling a constant, more powerful source with a weaker, (ideally) variable source.

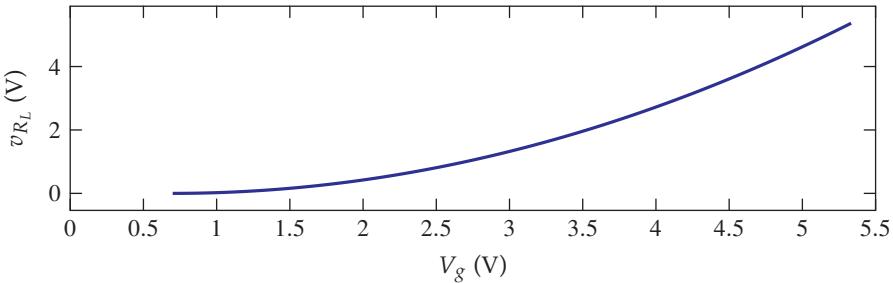


Figure 4.9



## 4.4 Operational Amplifiers

The **operational amplifier** (opamp) is the queen of analog electronic components. The opamp is a *four-port nonlinear* voltage-controlled voltage source, but it's so much more. Here are a few applications from the opamp highlight reel: summing two signals, subtracting two signals, amplifying a signal, integrating a signal, differentiating a signal, filtering a signal, isolating two subcircuits, generating periodic functions (e.g. sinusoids and square waves), and analog feedback control. Although they are nonlinear, in most applications a linear approximation is sufficiently accurate.

Figure 4.10 shows the circuit symbol for the opamp. Three terminals are displayed: **inverting input (-)** The **inverting input** is labeled with the “-” symbol. **non-inverting input (+)** The **non-inverting input** is labeled with the “+” symbol. **output** The **output** extends from the tip of the symbol, opposite the inputs.

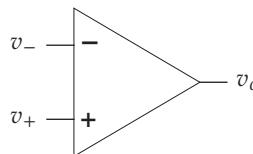


Figure 4.10. Circuit symbol for an opamp.

These comprise an input and an output port. However, there are two power supply ports that are typically suppressed in the circuit diagram. These two power supply ports are from a **differential supply**, which has a positive terminal (e.g. +12 V), symmetrically negative terminal (e.g. -12 V), and a common ground. The supply provides the opamp with external power, making it an **active** element.

When an opamp is operating in its linear mode, it outputs a voltage  $v_o$  that is  $A$  times the difference between its inputs  $v_+$  and  $v_-$ . The **open-loop gain**  $A$  is different for every opamp, but is usually greater than  $10^5$ . Let's formalize this model.

### Definition 4.5

An opamp's input terminals + and - draw zero current (i.e. have infinite input impedance). Let  $A$  be a positive real number. The output voltage  $v_o$  is given by

$$v_o = A(v_+ - v_-).$$

The output terminal has zero impedance.

Note that this model is equivalent to a **dependent voltage source** controlled by the input voltage difference. In fact, it is also *linearly* dependent, so linear circuit analysis techniques can be applied.<sup>5</sup>

The model is fairly accurate as long as  $|v_o|$  is less than the maximum power source voltage. Due to the high open-loop gain, the difference in input gain is highly restrictive for linear operation. This turns out not to be difficult to achieve, but does lead to a convenient approximation during analysis that applies most of the time:

$$v_+ \approx v_-$$

because other voltages in the circuit are typically much larger than the input voltage difference. We cannot, however, make this assumption unless (1) the opamp is operating in linear mode and (2) the opamp is part of a circuit that connects its output—via a wire or circuit elements—back to its inverting input (−).

This second condition is called **negative feedback** and is used in most opamp circuits for several reasons, the most important of which is that section 4.4 holds due to the virtual guarantee of linear operation in this case.

#### 4.4.1 Negative Feedback

We can think of negative feedback as continuously adjusting the output such that section 4.4 is approximately true.<sup>6</sup> Consider the feedback of  $v_o$  to the inverting input (called **unity feedback**), as shown in figure 4.11a, such that the output equation can be transformed as follows:

$$\begin{aligned} v_o &= A(v_+ - v_-) \\ &= A(v_i - v_o) \quad \Rightarrow \\ v_o &= \frac{A}{1+A}v_i \end{aligned}$$

Since  $A \gg 1$ ,  $v_o \approx v_i$ . In other words, for negative unity feedback,  $v_o$  follows  $v_i$ . For this reason, this particular opamp circuit is called a **voltage follower**. Let's consider negative feedback's effect on the difference in input voltage:

$$\begin{aligned} v_+ - v_- &= v_+ - \frac{A}{1+A}v_+ \\ &\approx 0. \end{aligned}$$

5. Note that, while the transistor can be considered a nonlinear dependent current source, the opamp can be considered a linear dependent voltage source. However, we can easily adapt an opamp circuit to behave as a linear dependent current source, so typically the opamp is still preferred.

6. Negative feedback is considered in detail in courses on control theory. The opamp was used extensively for feedback control until low-cost, high-performance digital microcontrollers became available. Opamp-based feedback control is now called *analog feedback control*, which still has certain applications.

This is equivalent to section 4.4. That is, *for negative feedback, the input voltages are nearly equal:  $v_+ \approx v_-$* . This is control theory—this is how we make a system behave the way we want! In this instance, the **loop gain**—the effective gain from  $v_i$  to  $v_o$ —is one. This same principle applies when elements such as resistors and capacitors are placed in the feedback path. The resulting loop gain can be nonunity and respond dynamically to the signal.

#### 4.4.2 Non-Inverting Opamp Circuit

The **non-inverting opamp circuit** is shown in figure 4.11b. Let's analyze the circuit to find  $v_o(v_i)$ . We begin with the KVL expression for  $v_o$  in terms of  $v_{R_1}$  and  $v_{R_2}$ :

$$v_o = v_{R_1} + v_{R_2}.$$

Let's use Ohm's law to write:

$$v_o = i_{R_1} R_1 + i_{R_2} R_2.$$

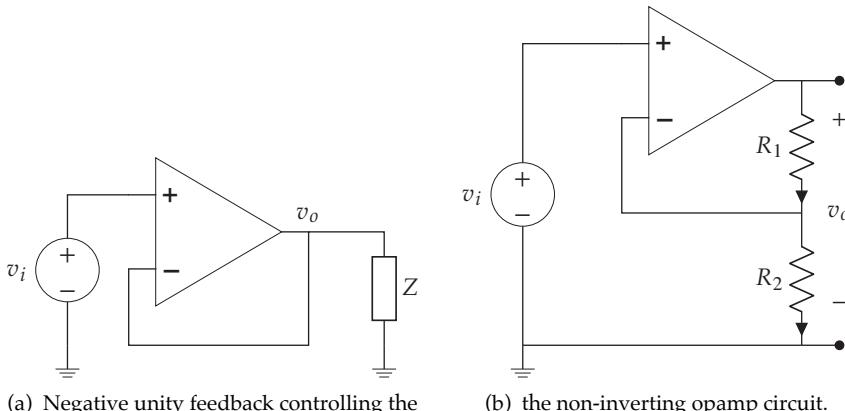


Figure 4.11. Two opamp circuits.

The KCL equation for the node between  $R_1$  and  $R_2$  gives

$$\begin{aligned}
 i_{R_1} - i_{R_2} - i_{\text{in}} &= 0 \quad \Rightarrow \\
 i_{R_1} &= i_{R_2} \quad \Rightarrow \\
 v_o &= i_{R_2}(R_1 + R_2) \quad \Rightarrow \\
 i_{R_2} &= v_o / (R_1 + R_2).
 \end{aligned}$$

We can write another equation for  $v_o$  from the opamp:

$$\begin{aligned} v_o &= A(v_+ - v_-) \\ &= A(v_i - v_{R_2}). \end{aligned}$$

We have an expression for  $i_{R_2}$  that can eliminate  $v_{R_2}$  with a little Ohm's law action:

$$\begin{aligned} v_o &= A(v_i - i_{R_2}R_2) \\ &= A(v_i - v_oR_2/(R_1 + R_2)) \Rightarrow \\ v_o + v_oAR_2/(R_1 + R_2) &= Av_i \\ v_o &= \frac{A}{1 + AR_2/(R_1 + R_2)}v_i \\ &= \frac{(R_1 + R_2)/R_2}{(R_1 + R_2)/(R_2K) + 1}v_i. \end{aligned}$$

If  $A \gg (R_1 + R_2)/R_2$ , the denominator of this expression goes to 1 and we have the loop gain approximately

$$\frac{R_1 + R_2}{R_2}.$$

This gives the following input-output equation for the circuit:

$$v_o \gg \frac{R_1 + R_2}{R_2}v_i. \quad (4.3)$$

It is highly significant that equation (4.3) doesn't depend on  $A$ , which can be quite variable. Rather, it depends on the resistances  $R_1$  and  $R_2$ , only—and these are very reliable. As long as the condition

$$A \gg \frac{R_1 + R_2}{R_2}$$

is satisfied, equation (4.3) is valid.

This independence of the input-output relationship on the open-loop gain  $A$  is very common for opamp circuits. We have essentially traded gain for better linearity and gain invariance. It can be shown that this is equivalent to the assumption that  $v_+ \approx v_-$ . Making this assumption earlier in the analysis can simplify the process. Note that we do not use the assumption for the opamp equation  $v_o = A(v_+ - v_-)$ , for this would imply  $v_o = 0$ . Instead, in the previous analysis, we could immediately assume that  $v_{R_2} = v_i$  and proceed in a similar fashion.

---

## 4.5 Problems



**Problem 4.1**  **RHINOCEROS** Write a one- or two-sentence response to each of the following questions and imperatives. The use of equations is acceptable when they appear in a sentence. Don't quote me (use your own words, other than technical terminology).

- Write the equivalent impedance of a resistor  $R$  and an inductor  $L$  in series. Express the result in rectangular and polar (phasor) form.
- How do you find the Norton equivalent resistance?
- Explain how a diode operates in forward-bias.
- In a MOSFET, how much current will flow from the drain  $D$  to the source  $S$  when the gate-source voltage is 0.3 V? Succinctly explain/justify.

**Problem 4.2**  **FLAMINGO** Write a one- or two-sentence response to each of the following questions and imperatives. The use of equations is acceptable when they appear in a sentence. Don't quote me (use your own words, other than technical terminology).

- Describe a couple differences between MOSFETs and opamps.
- If a DC source is connected to a circuit in steady state, describe how an inductor in the circuit will be operating.
- If a transformer increases an AC signal's voltage by a factor of 119, what happens to the signal's current?
- How do we determine the diode resistance for the piecewise linear model of a diode?

**Problem 4.3**  **ASTRINGENT** Write a one- or two-sentence response to each of the following questions and imperatives. The use of equations is acceptable when they appear in a sentence. Don't quote me (use your own words, other than technical terminology).

- If the current through an inductor is suddenly switched off, what happens?
- Let the output voltage of a resistor circuit be 5 V and the equivalent resistance  $500\ \Omega$ . What is the Thevenin equivalent circuit?
- In the preceding part of this question, what is the Norton equivalent?
- When can we use impedance analysis?

**Problem 4.4**  **PROLONGATE** For the circuit diagram of figure 4.12, solve for  $v_o(t)$  if  $V_s(t) = A \cos \omega t$ . Let  $N = n_2/n_1$ , where  $n_1$  and  $n_2$  are the number of turns in each coil, 1 and 2, respectively. Also let  $i_L(0) = 0$  be the initial condition.

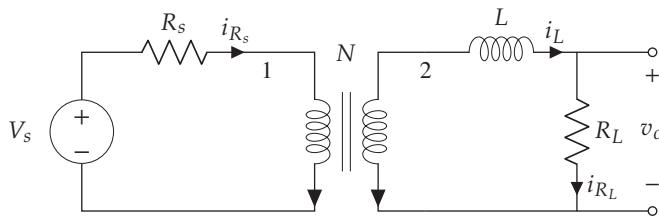


Figure 4.12. circuit diagram for problem 4.4 and problem 4.5.

**Problem 4.5** **SYNOPSES** Re-do Exercise 4.4, but only consider the steady-state response. Use impedance methods!

**Problem 4.6** **SPARTANISM** When considering the steady state of circuits with only DC sources, all voltages and currents are constant and all diodes are in constant states (each is ON or OFF). The methods of section 4.2 still apply, of course, but we needn't be concerned with a time evolution.

Consider the circuits of figure 4.13. For each circuit, solve for the voltage across the  $5\text{ k}\Omega$  resistor. Treat each diode as an ideal diode.

**Problem 4.7** **OUTSMART** Repeat Exercise 4.6, but use the piecewise linear model of each diode.

**Problem 4.8** **COMBMAKER** A **diode clipping circuit** is one that “clips” the tops and or bottoms of a signal. These circuits can be used to set a maximum or minimum voltage for a signal.

Consider the diode clipping circuit of figure 4.14. Source  $V_1$  effectively adjusts the *maximum* possible load voltage  $v_{R_L}$ , and  $V_2$  the *minimum*. Let  $V_S(t) = 10 \cos 4\pi t$ ,  $V_1 = 5\text{ V}$ ,  $V_2 = -3\text{ V}$ , and  $R_s = R_L = 50\text{ }\Omega$ . Solve for  $v_{R_L}(t)$ . Use the ideal diode model.

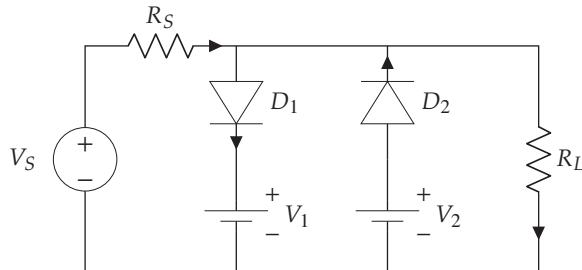


Figure 4.14. A diode clipping circuit for Exercise 4.8.

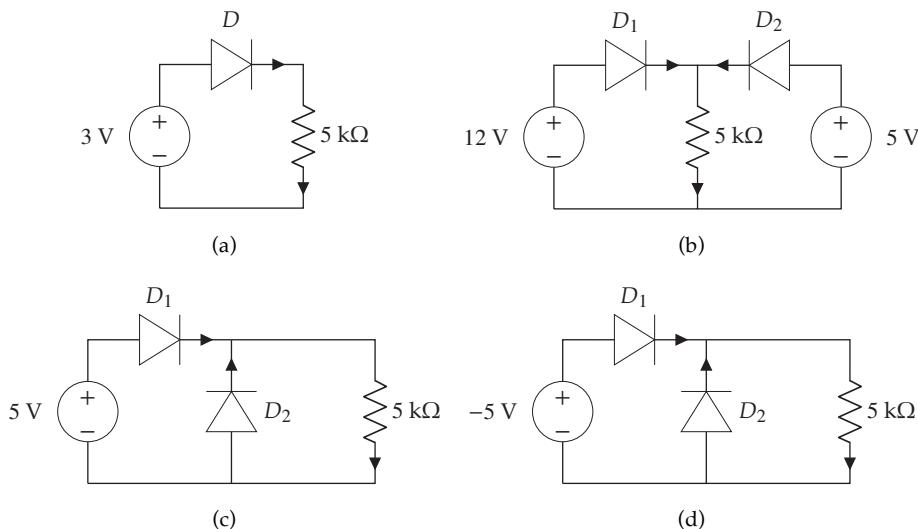


Figure 4.13. Diode circuits for Exercise 4.6.

**Problem 4.9** **CLOISTERAL** Repeat problem 4.8, but use the piecewise linear model of each diode.

**Problem 4.10** **DIASPORA** For the circuit diagram of figure 4.15, solve for  $v_o(t)$  if  $V_s(t) = A$  for some given  $A > 0.6$  V. Let  $v_C(t)|_{t=0} = 0$  V be the initial condition. Use a piecewise linear model for the diode with some  $R_d \in \mathbb{R}_{\geq 0}$ . Do not estimate  $R_d$ .

**Problem 4.11** **POROSITY** For the circuit shown in figure 4.16, determine the voltage across the load  $v_{R_L}$  in terms of parameters and the gate voltage source voltage  $V_g$  and  $V_s$ . The parameters of the MOSFET are  $K$  and  $V_T$ . Assume MOSFET saturation operation.

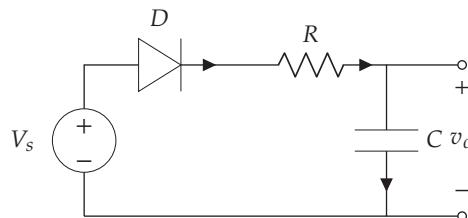


Figure 4.15. Circuit diagram for problem 4.10.

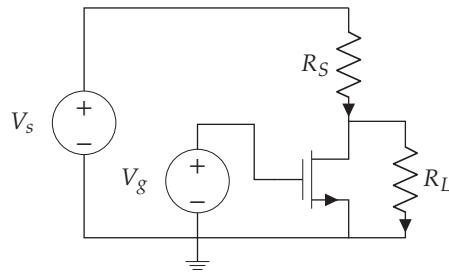


Figure 4.16. Circuit for Exercise 4.11.

**Problem 4.12** **OVERBROIL** The opamp circuit of figure 4.17 is used as a voltage-controlled current source for the load  $R_L$ . Show that it behaves as a current source with current  $i_{R_L}$  controlled by voltage source  $v_i$ .

Use two separate methods: (a) assuming  $v_+ \approx v_-$  and (b) not assuming  $v_+ \approx v_-$ , rather, assuming the open loop gain of the opamp  $A$  is large. Comment on the differences between the methods of (a) and (b).

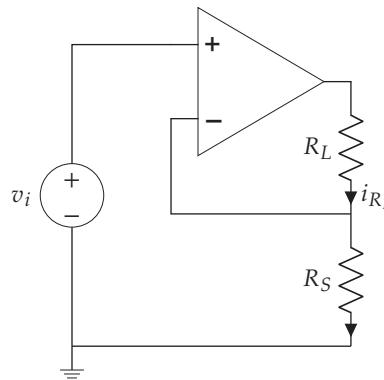


Figure 4.17. Circuit for Exercise 4.12.

**Problem 4.13** **POLYNUCLEATE** Use the circuit diagram of figure 4.18 to answer the questions below. Use the sign convention from the diagram. Let  $v_i = A \cos \omega t$  be an ac input voltage. The load  $Z_L$  impedance is not given.

- Write the elemental equations in terms of  $Z_{R_1}$ ,  $Z_{R_2}$ ,  $Z_{R_S}$  and  $Z_L$  (the impedances of the components).
- Write the KCL and KVL equations.

(c) Solve for the steady-state  $v_o(t)$  without inserting the values of the impedances (that is, leave it in terms of  $Z_{R_1}$ ,  $Z_{R_2}$ ,  $Z_{R_S}$  and  $Z_L$ ).

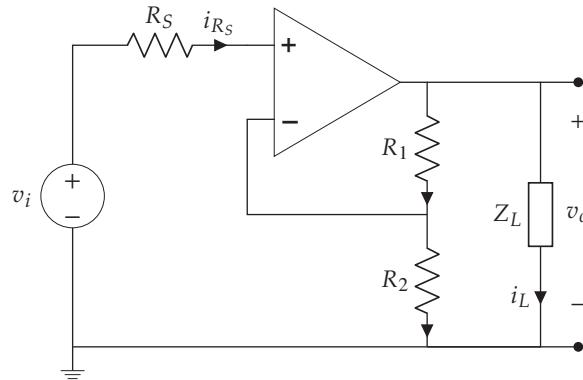


Figure 4.18. Circuit for Exercise 4.13.

**Problem 4.14** **LUSH** Consider the circuit in figure 4.19. Solve for  $v_o(t)$  for input voltage  $v_i(t) = 5$  V, a sine wave of  $v_i(t) = 5 \sin 25t$ , and a sine wave of  $v_i(t) = 5 \sin 2525t$ . Let  $R_1 = 50 \Omega$ ,  $R_2 = 10 \text{ k}\Omega$ ,  $C = 10 \mu\text{F}$ , and the opamp open-loop gain be  $A = 10^5$ . Let the initial condition be  $v_C(t) = 0$  V. In each case, plot the solution to show the transient response until it reaches steady-state.

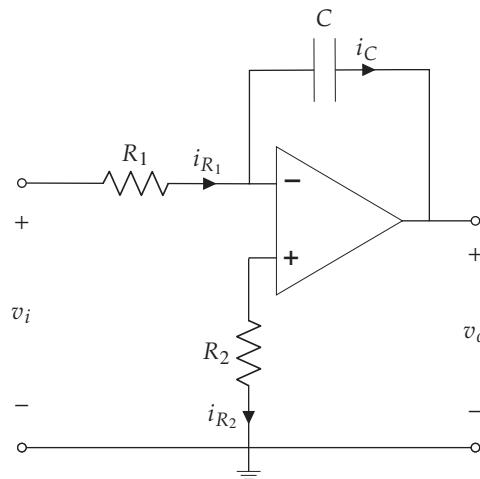


Figure 4.19. Opamp circuit for Exercise 4.14

**Problem 4.15**  **HOGWASH** Consider the circuit in figure 4.20. Solve for  $v_o(t)$  for a known input voltage  $v_i(t)$ .

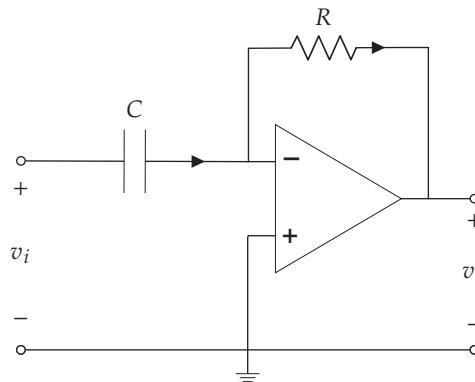


Figure 4.20. Opamp circuit for Exercise 4.15

**Problem 4.16**  **VIRTUE** In each of the figures of figure 4.21, solve for the voltage  $v_{100}$  across the  $100\ \Omega$  resistor. Use the assumptions in the associated caption. Clearly justify each response.

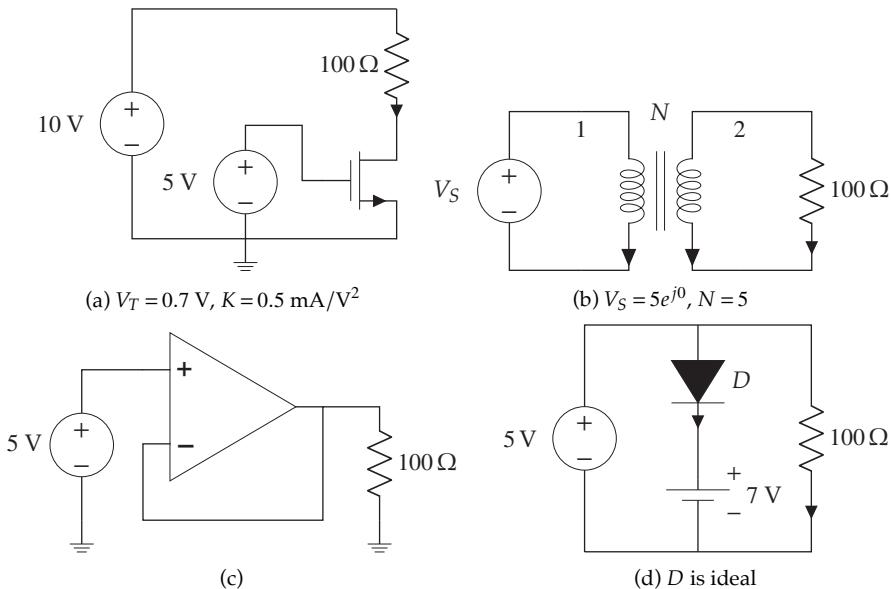
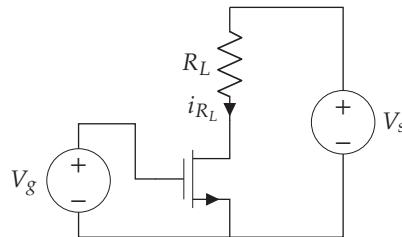
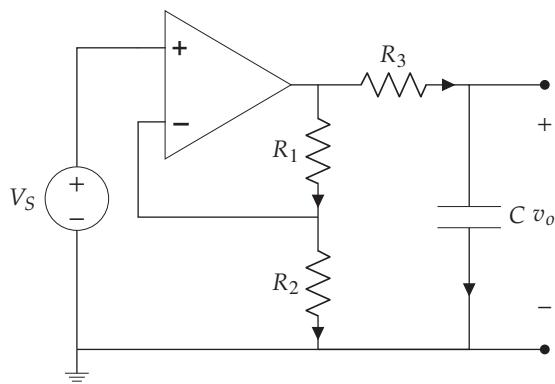


Figure 4.21. Circuits for problem 4.16.

**Problem 4.17** Consider the circuit below with input voltage sources  $V_S$  and  $V_g$ . Determine  $V_g$  such that the load voltage  $v_{RL} = 10$  V. Let  $R_L = 2$  kΩ,  $K = 0.5 \text{ mA/V}^2$ ,  $V_T = 0.7$  V,  $V_s = 20$  V.



**Problem 4.18** Consider the circuit below with input voltage source  $V_S(t) = A$  where  $A > 0$  is a known (but unspecified) constant. Perform a circuit analysis to solve for  $v_o(t)$  for the initial condition  $v_C(0) = 0$ . Hint: it is easier if you realize the opamp output voltage is effectively an ideal voltage source (so it does *not depend* on  $v_{R_3}$  and  $v_C$ ) and you can therefore treat the two parts of the circuit separately.



**Problem 4.19** **SATISFIED** In each of the figures of figure 4.22, solve for the voltage  $v_{1k}$  across the  $1\text{ k}\Omega$  resistor. Use the assumptions in the associated caption. Clearly justify each response.

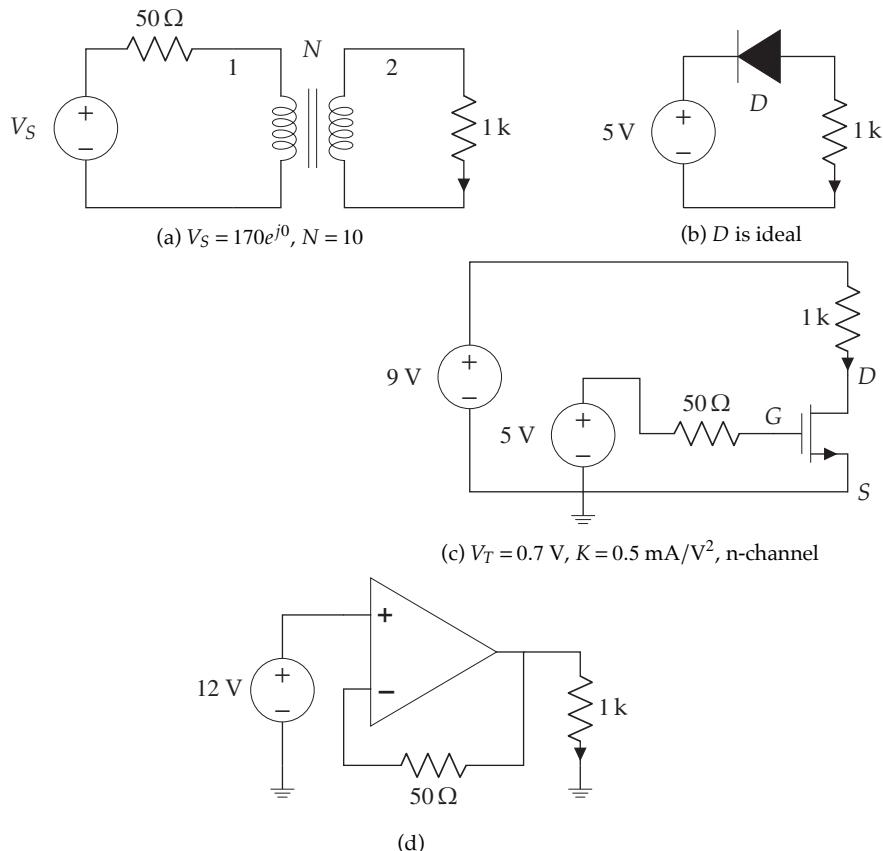


Figure 4.22. Circuits for problem 4.19.

**Problem 4.20** Write a one- or two-sentence response to each of the following questions and imperatives. The use of equations is acceptable when they appear in a sentence. Don't quote me (use your own words, other than technical terminology).

- Give an example of an application of a transformer and explain how the transformer functions for this application.
- Let the output current of a resistor circuit be 1 A and the equivalent resistance  $100\Omega$ . What is the Norton equivalent circuit?
- In the preceding part of this question, what is the Thevenin equivalent?
- If a wire is connected between the terminals of a battery, what happens?



## Bibliography

Agarwal, A., and J. Lang. 2005. *Foundations of Analog and Digital Electronic Circuits*. The Morgan Kaufmann Series in Computer Architecture and Design. Elsevier Science.

Horowitz, P., and W. Hill. 2015. *The Art of Electronics*. Cambridge University Press.

Ulaby, Fawwaz T., Michel M. Mahabiz, and Cynthia M. Furse. 2018. *Circuit Analysis and Design*. ISBN 978-1-60785-484-5. Michigan Publishing.



## Contributors

**Associate Professor Rico A. R. Picone**  
Department of Mechanical Engineering  
Saint Martin's University  
Lacey, Washington, USA